MA 262 Homework 6: Curl up and Div

Do these problems concerning divergence and curl.

pg 221: 1, 3, 7, 8, 12

Problem 5.A Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$.

(1) Find the scalar curl of $\mathbf{F}$ using the partial derivative formula. (The computation is a little messy, but the answer is pretty.)

(2) Let C be the unit circle centered at the origin, oriented counter-clockwise. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.

(3) Give two reasons why $\mathbf{F}$ is not conservative.

Problem 5.B Let

$$\mathbf{F}(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

for fixed values of a,b,c,d.

(1) Find the divergence of $\mathbf{F}$ using the partial derivative formula.

(2) Find the curl of $\mathbf{F}$ using the partial derivative formula.

(Those of you who have taken linear algebra may recognized that the divergence of a linear vector field is the trace.)

More on Conservative Vector Fields

It turns out (via Green's theorem) that there is a very close connection between curl and gradient fields. These problems are intended to remind you of some basic facts about gradient fields. You don't need anything about curl to do them, but you will need to use the Fundamental Theorem of Calculus for conservative vector fields.

pg 399: (do not use curl for any of these problems) 1, 2, 7

Problem 5.C We know from the FTC for Conservative Vector Fields that Conservative Vector Fields have path independent line integrals. The point of this problem is to get us strated on a proof of the converse:

Theorem. Suppose that $\mathbf{F}$ is a vector field defined on an open set $U \subset \mathbb{R}^2$. Suppose that whenever $\mathbf{x}\colon [a,b] \to U$ and $\mathbf{y}\colon [a',b'] \to U$ are piecewise $\mathbf{C}^1$ paths such that $\mathbf{x}(a) = \mathbf{y}(a')$ and $\mathbf{x}(b) = \mathbf{y}(b')$ then

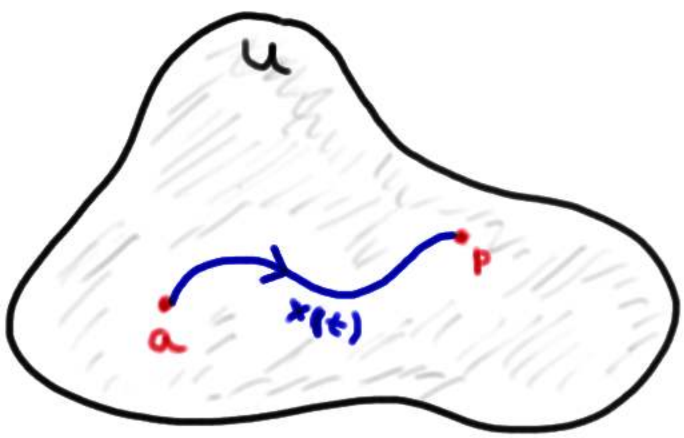
$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}.$$

Then $\mathbf{F}$ is a gradient field.

Remark: The assumption in the theorem is merely saying that the line integral of $\mathbf{F}$ over a curve depends only on the endpoints of the curve and not on the actual path taken.

Do the following to prove the theorem:

(1) We begin by defining a function f that we will later prove to be a potential function. To define f, fix a point $\mathbf{a} \in U$. For any $\mathbf{p} \in U$ define $f(\mathbf{p}) = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{x}$ is a path from $\mathbf{a}$ to $\mathbf{p}$. Explain why the value of f depends only on $\mathbf{F}$, $\mathbf{a}$, $\mathbf{p}$, and not on chosen path from $\mathbf{a}$ to $\mathbf{p}$. See the figure below:



(2) For the function f defined in the previous part, explain how the values of f change if a point $\mathbf{b} \neq \mathbf{a}$ is used instead of $\mathbf{a}$.

In class, we'll actually prove that $\nabla f = \mathbf{F}$. The proof will involve applying the definition of the derivative from Calc 1.

Some Interesting Vector Fields

The relationship between vector fields and the region where they are defined is a very interesting one. It turns out that the topological type (eg. the number of holes) of the region has a big impact on the sort of vector fields that are defined on it. The next two problems provide some indication of the relationship.

Problem 5.D Suppose that $U \subset \mathbb{R}^2$ is an open set and that $\mathbf{F}$ and $\mathbf{G}$ are two C^1 vector fields defined on U. Let α and β be two real numbers.

(1) Prove that if $\mathbf{F}$ and $\mathbf{G}$ is a conservative vector field, then so is the vector field $\alpha\mathbf{F}+\beta\mathbf{G}$. (Hint: think about what happens if you add and scale potential functions.)

(2) Suppose that $\mathbf{F}$ and $\mathbf{G}$ both have scurl equal to 0 everywhere. Explain why $\alpha\mathbf{F}+\beta\mathbf{G}$ also has scurl equal to zero everywhere.

Problem 5.E Let $\mathbf{F}_0(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Recall that in Problem 5.A you showed that $\mathbf{F}_0$ is not conservative but does have curl everywhere equal to zero. Notice also that $\mathbf{F}_0$ is defined everywhere except at the origin.

(1) Let $\mathbf{F}_1(x,y) = \mathbf{F}_0(x-1,y)$. Explain why $\mathbf{F}_1$ is a vector field defined everywhere except at $(1,0)$ which has scalar curl everywhere equal to zero, but is not conservative. (Hint: Don't do this by doing a calculation– think geometrically.)

(2) Give an example of a vector field which is defined everywhere except at $(0,0)$ and $(1,0)$, is not conservative and has scalar curl equal to 0. (Hint: Use Problem 5.D) Can you find infinitely many such vector fields?

(3) Explain why for any n points $\mathbf{p}_1,\dots,\mathbf{p}_n$ in $\mathbb{R}^2$ there is a vector field that is defined everywhere except at $\mathbf{p}_1,\dots\mathbf{p}_n$, has curl zero, and is not a gradient field. Can you find infinitely many such vector fields? Can you find a different such a vector field for each vector $(\alpha_1,\alpha_2,\dots,\alpha_n) \in \mathbb{R}^n$? (Hint: Use Problem 5.D)

Informally, the last problem says that if a region has n holes, then the space of non-conservative vector fields with scalar curl everywhere zero is at least n. In a future homework assignment you'll show that it is exactly n.