MA 262 Homework 4: Integrate your life!

The ultimate goal of this course is to understand the relationship between vector fields, scalar fields, paramterized curves, and parameterized surfaces. The goal of this homework assignment is to get you comfortable with parameterized curves. In a future homework assignment, we'll elaborate on some of the problems below.

Read Sections 3.3, and 6.1. As you read, pay attention to:

- How to integrate a scalar field over a parameterized curve.
- How to integrate a vector field over a parameterized curve.

Problem 4.A If $\mathbf{x}\colon [a,b] \to \mathbb{R}^2$ is a C^1 curve with non-zero speed, we define the **total curvature** of $\mathbf{x}$ to be the integral

$$\int_a^b ||\mathbf{T}'(t)||dt.$$

(1) Show that the total curvature of a circle of radius r is 2π.

(2) Let $r > 0$ and $k > 0$ be constants. Show that the total curvature of the path $\mathbf{x}(t) = (r\cos t, r\sin t, kt)$ for $0 \leq t \leq 2\pi$ is $2\pi r/\sqrt{r^2+k^2}$. Notice the difference between the total curvature of the helix and the total curvature of the spiral. What happens to the total curvature of the helix as k increases or decreases? Make a connection to the total curvature of the circle.

(3) Show that if $\mathbf{x}$ is a parameterization by arc length, then the total curvature is $\int_0^L \kappa(t)\,dt$ where L is the length of the curve and κ is the curvature of $\mathbf{x}$.

(4) Prove that total curvature does not depend on the parameterization of the curve. That is, if $h\colon [c,d] \to [a,b]$ is a change of coordinates function then the total curvature of $\mathbf{y}(t) = \mathbf{x}(h(t))$ is the same as the total curvature of $\mathbf{x}$.

Hint: You need to use the chain rule when calculating $\mathbf{y}'$. When you integrate, you'll need to use substitution.

Remark: A theorem of Fenchel states that the total curvature of any closed curve in the plane is at least 2π. At the age of 18, John Milnor proved that the total curvature of any knotted closed curve in $\mathbb{R}^3$ is at least 4π.

Do these problems from 6.1: pg 379: 1 - 9, 11, 25

These problems give you practice using the definitions of line integral. You may find it helpful to review the notion of "vector field" from the first day of class.

On problem 11, remember that if $\mathbf{x}(t) = (x(t), y(t))$, then by definition $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$. The problem is just asking for the path integral of the vector field $\vec{F}(x,y) = (-y,x)$.

Problem 25: If you don't know the legend, you may like to read up on Sisyphus on Wikipedia (for example). My favorite pop-culture reference to Sisyphus is in the movie "Party Girl". You may assume that the force due to gravity is $\mathbf{F}(x,y,z) = (0,0,-75)$ since pounds is a unit of force.

Do these problems from 3.3. These problems give you practice sketching and understanding vector fields. Remember that you can use Grapher to draw vector fields (including those in problems 1 - 13)!

pg 213: 1 - 5, 9, 13