MA 262 Homework 3: If you get an undeserved speeding ticket, is it a moving frame?

The ultimate goal of this course is to understand the relationship between vector fields, scalar fields, paramterized curves, and parameterized surfaces. The goal of this homework assignment is to get you comfortable with parameterized curves. In a future homework assignment, we'll elaborate on some of the problems below.

Read Sections 3.1 and 3.2

We'll cover 3.2 in a fair amount of detail in class. As you read pay attention to:

- The definition of velocity, speed, and acceleration of a path.
- The definition of length of a path. Why is it a sensible definition?
- How to reparameterize by arc length. This is a concept many students find challenging – take the time to absorb it.
- The definition of unit tangent vector and curvature

Associated to (most) every path are a number of intrinsic geometric quantities. These include length, curvature, and the moving frame. These problems ask you to calculate some of these quantities. Unfortunately, most of the time these quantities are extremely hard to calculate (although approximations are easy to get). This means that you can only be asked for relatively simple examples (but even those examples can be a challenge!)

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Problem 3.A: Find integrals in 1 variable equal to the lengths of the following curves. You do not need to solve the integrals.

(1) $f(t) = (t, t^2, t^3)$ for $-1 \leq t \leq 1$.
(2) $g(t) = (\sqrt{t}, \cos t)$ for $t \in [2, 7]$.

Problem 3.B: Let $\mathbf{x}(t) = (t^3, 2t^3 + 2)$ for $1 \leq t \leq 2$. Reparameterize $\mathbf{x}$ by arc-length.

Problem 3.C: Let $\mathbf{x}\colon [0,b] \to \mathbb{R}^2$ be a parameterized curve. In class, we learned how to reparameterize $\mathbf{x}$ to become a curve $\mathbf{y}$ so that on the interval $[0,t]$, the curve $\mathbf{y}$ has length t. Let $k > 0$ be constant. Show how to reparameterize $\mathbf{x}$ to a curve $\mathbf{z}$ so that on the interval $[0,t]$ the length of $\mathbf{z}$ is kt.

Hint: Start by reparameterizing $\vec{x}$ to $\mathbf{y}$. How do you need to scale the input to $\mathbf{y}$ so that instead of "distance equalling time", "distance equals k times time"?

Problem 3.D: Let G be the graph of a function $y = f(x)$ in the plane (with f differentiable). A circle of radius ρ is on top of and tangent to G and rolling along it (from left to right) at a rate of 1 revolution per second. At time 0, the circle is tangent to $(0, f(0)$. Let P be a point on the circle. (Which point you choose is up to you.) The purpose of this problem is to find a parameterization $\mathbf{p}(t)$ for the path taken by P.

You will not be able to find a "closed form" solution – your answer will likely have inverse functions and possibly integrals in it (at least implicitly). But, however you write your answer the reader should, in principle, be able to trace everything back to the function f. The problem is essentially asking for a recipe for how to figure out a parameterization of $\mathbf{p}(t)$. You don't need to actually follow the recipe, just give me enough details so that if I knew what f was, I could follow your recipe. Capice?

(1) Find a formula for the position $\mathbf{p}(t)$ of P in tangent space coordinates based at the center $\mathbf{c}(t)$ of the circle.

(2) After t seconds, how far along G has the circle rolled?

(3) Find a parameterization $\mathbf{x}$ of G so that at time t, the circle is tangent to $\mathbf{x}(t)$. (Hint: Use problem C above)

(4) Write $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Find a vector perpendicular to $\mathbf{x}'(t)$ that (when based at $\mathbf{x}(t)$) points above G (rather than below). (Your answer will depend on x and y.)

(5) Find a formula for the center $\mathbf{c}(t)$

(6) Find coordinates for $\mathbf{p}(t)$ in the usual coordinate system. (Your answer will likely have an inverse function in the expression – you won't be able to get it in closed form.)