MA 302: Practice Exam 3 **Name:**________________________

These questions cover only the material since Exam 2. The first few problems are repeated from the previous practice exam.

(1) Find a parameterization of the surface formed by the graph of $z = x^2 - y^2$ with (x,y) in the triangle in the xy-plane formed by the x-axis, the y-axis, and the line $y = -x+1$.

Solution: How about:

$$\mathbf{X}(s,t) = \begin{pmatrix} s \\ t \\ s^2 - t^2 \end{pmatrix}$$

with $0 \leq s \leq 1$ and $0 \leq t \leq -s+1$?

(2) Is the surface in the previous problem a smooth surface? If no, at what points is it not smooth?

Solution: The answer depends (somewhat) on your parameterization. The answer here is based on the parameterization above.

You can calculate that

$$\begin{aligned} \mathbf{T}_s &= (1,0,2s) \\ \mathbf{T}_t &= (0,1,-2t) \\ \mathbf{N} &= (-2s,2t,1) \end{aligned}$$

Since $\mathbf{N}$ is never $\mathbf{0}$, and since $\mathbf{X}$ is obviously C^1, $\mathbf{X}$ is a smooth surface.

Solution: How about

$$\mathbf{X}(s,t) = \begin{pmatrix} \cos s(\cos t + 5) \\ 2\sin t \\ \sin s(\cos t + 5) \end{pmatrix}$$

for $0 \leq t \leq 2\pi$ and $0 \leq s \leq 2\pi$?

(3) Consider the surface

$$\mathbf{X}(s,t) = \begin{pmatrix} 2\sin 3t + t \\ \cos 2s \\ t^2 + s^2 \end{pmatrix}, \quad 0 \leq t \leq \pi/4, \quad 0 \leq s \leq \pi$$

Find the tangent and normal vectors to $\mathbf{X}$ at the point $(\pi/6, \pi/6)$. Is the surface smooth?

Solution:

We have

$$\begin{aligned}
\mathbf{T}_s &= (0, -2\sin 2s, 2s) \\
\mathbf{T}_t &= (6\cos(3t)+1, 0, 2t) \\
\mathbf{N} &= (-4t\sin 2s, 2s(6\cos 3t+1), 2\sin 2s(6\cos 3t+1)
\end{aligned}$$

Plug $(\pi/6, \pi/6)$ into the above equations to get:

$$\begin{aligned}
\mathbf{T}_s &= (0, -\sqrt{3}, \pi/3) \\
\mathbf{T}_t &= (1, 0, \pi/3) \\
\mathbf{N} &= (-\pi\sqrt{3}/3, \pi/3, \sqrt{3})
\end{aligned}$$

Since $\mathbf{N}(\pi/6, \pi/6) \neq \mathbf{0}$, the surface is smooth at that point.

(4) Let S be the disc of radius 1 centered at $(1,0,0)$ in $\mathbb{R}^3$ which is parallel to the yz-plane. Orient S with normal vector pointing in the direction of the postive x-axis. Use the definition of surface integral to calculate the flux of $\mathbf{F}(x,y,z) = (-xy, yz, xz)$ through S.

Solution: Parameterize S as:

$$\mathbf{X}(s,t) = \begin{pmatrix} 1 \\ s \\ t \end{pmatrix}$$

with (s,t) in the region D defined by $0 \leq s^2 + t^2 \leq 1$. It is easy to calculate $\mathbf{N} = (1,0,0)$. Then,

$$\mathbf{F} \cdot \mathbf{N}(x,y,z) = -xy.$$

Thus, by the definition of surface integral, the flux of $\mathbf{F}$ through S is

$$\iint_D \mathbf{F} \cdot \mathbf{N}(\mathbf{X}(s,t))\, dA = \iint_D -s\, ds\, dt.$$

Change to polar coordinates by setting $s = r\cos\theta$ and $t = r\sin\theta$. Then the integral above is equal to (by the change of coordinates theorem):

$$\int_0^1 \int_0^{2\pi} -r^2 \cos\theta\, d\theta dr$$

Since $\int_0^{2\pi} \cos\theta d\theta = 0$, the flux equals 0.

(5) Use the same surface S and $\mathbf{F}$ as in the previous problem, but now use Stoke's theorem to calculate the flux of the curl from the previous problem.

Solution: By Stoke's theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\, d\mathbf{s}.$$

Parameterize ∂S as:

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}$$

with $0 \leq t \leq 2\pi$.

Notice that $\mathbf{x}$ gives ∂S the orientation induced by the orientation on S. Then,

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x})(t) \cdot \mathbf{x}'(t)\, dt.$$

Calculations show that this equals

$$\begin{aligned} \int_0^{2\pi} -\cos t \sin^2 t + \sin t \cos t\, dt &= \int_0^{2\pi} -\cos t \sin^2 t\, dt + \int_0^{2\pi} \sin t \cos t\, dt \\ &= 0. \end{aligned}$$

(6) Give precise statements of Stokes' Theorem and the Divergence Theorem.

Stokes' Theorem: Let $S \subset \mathbb{R}^3$ be a compact oriented piecewise C^1 surface such that ∂S is piecewise C^1. Give ∂S the orientation induced by S. Suppose that $\mathbf{F}$ is a C^1 vector field defined on S. Then

$$\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int\limits_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Divergence Theorem Suppose that $V \subset \mathbb{R}^3$ is a compact 3-dimensional region with piecewise C^1 boundary. Give ∂V the orientation with outward pointing normal. If $\mathbf{F}$ is a C^1 vector field defined throughout V, then

$$\iiint\limits_V \operatorname{div} \mathbf{F}\, dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S}.$$

(7) State and prove Gauss' law for gravity.

Solution: See the course notes.

(8) Use Gauss' Law for gravity and symmetry considerations to prove the shell theorem.

Solution: See the course notes.

(9) Suppose that a vector field $\mathbf{F}$ defined on $\mathbb{R}^3 - \{\mathbf{0}\}$ has a flux of 21 through a sphere of radius 2 (oriented outward). If the divergence of $\mathbf{F}$ is a constant -1, what is the flux of $\mathbf{F}$ through a sphere of radius 4 (oriented outward)?

Solution: Let S_2 be the sphere of radius 2 and let S_4 be the sphere of radius 4. Let V be the region between them. Notice that $\mathbf{F}$ is C^1 throughout V. If we give ∂V the outward pointing orientation, then S_2 is oriented "the wrong way". Thus, by the divergence theorem:

$$-\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{F} \, dV.$$

Since $\operatorname{div} \mathbf{F} = -1$, the last integral is just the negative of the volume of V. The volume of V is $\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(2)^3 = 64\pi$. Thus,

$$-\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = -64$$

Since the flux through S_2 is 21, we have

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = -64 + 21 = -43.$$

(10) Suppose that $\mathbf{F}$ is a C^1 vector field that is everywhere tangent to the unit sphere in $\mathbb{R}^3$. Explain why the flux of $\mathbf{F}$ through the sphere must be zero. If $\mathbf{F}$ is also C^1 everywhere inside the sphere, what can you conclude about the divergence of $\mathbf{F}$ inside the sphere?

Solution: Let $\mathbf{n}$ be the unit normal to the unit sphere S. We have:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

This equals 0, since $\mathbf{F}$ and $\mathbf{n}$ are always perpendicular on the sphere.

By the divergence theorem, if $\mathbf{F}$ is C^1 inside the sphere, then the integral of the divergence of $\mathbf{F}$ over the unit ball is equal to the flux of $\mathbf{F}$ across S, which we just calculated to be zero. Thus, integrating the divergence of $\mathbf{F}$ over the unit ball gives zero.

(11) Suppose that $\mathbf{F}$ is a C^1 vector field and that S is a compact surface without boundary. If the circulation of $\mathbf{F}$ around S is non-zero, what can you conclude about S?

Solution: S must be non-orientable. If it were orientable we could apply Stokes' theorem to conclude:

$$0 \neq \iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Since $\partial S = \varnothing$, this last integral must be zero.

(12) Suppose that two surfaces S_1 and S_2 have the same oriented boundary and that they are disjoint except along their boundaries. Suppose that $\mathbf{F}$ is a C^1 vector field defined on the region bounded by the union of S_1 and S_2. Explain why the circulation of $\mathbf{F}$ is the same on S_1 and S_2. If the vector field is incompressible, explain why the flux through S_1 is the same as the flux through S_2.

Solution: By Stokes theorem, the circulation of $\mathbf{F}$ on each S_i is equal to the circulation of $\mathbf{F}$ around the boundary. Since they have the same oriented boundary, they must have the same circulation.

For $\mathbf{F}$ to be incompressible, means that $\operatorname{div} \mathbf{F} = 0$. Let S be the union of S_1 and S_2 without outward normal. Let V be the region bounded by S. One of S_1 or S_2 has the wrong orientation (since they induce the same orienation on their common boundary). Thus by the divergence theorem:

$$0 = \iiint\limits_V \operatorname{div} \mathbf{F}\, dV = \iint\limits_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \pm \iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} \mp \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

Thus, the fluxes are the same.