MA 302: Practice Exam 2 **Name:**________________________

(1) Give an example of a vector field $\mathbf{F}$ having $\operatorname{curl}\mathbf{F} = \mathbf{0}$, but where $\mathbf{F}$ is not a gradient field.

(2) Know the formal definitions of the following terms or the complete precise statements of the following theorems:

(a) Green's theorem

(b) planar divergence theorem

(c) Stokes' theorem

(d) Poincaré's theorem

(e) parameterized surface

(f) orientable surface

(g) one-sided surface

(3) Give an example of a one-sided surface in $\mathbb{R}^3$.

(4) Give an example of an orientable surface in $\mathbb{R}^3$.

(5) Be able to do the following:

(a) Suppose that $D \subset \mathbb{R}^2$ is the union of two "nice" regions D_1 and D_2 along an edge C in their boundaries. Suppose that $\mathbf{F}$ is a C^1 vector field defined on the union $D = D_1 \cup D_2$. Prove that $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{s}$.

(b) Give an outline of the proof of Green's theorem.

(c) Suppose that $X \subset \mathbb{R}^2$ is a simply connected open subset and that $\mathbf{F}\colon X \to \mathbb{R}^2$ has $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Prove that if C is a simple closed curve in X then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$. Use this to prove that $\mathbf{F}$ is conservative.

(6) Let $D \subset \mathbb{R}^2$ be the region bounded by the graphs of the equations $y = x^3$ and $y = x$ and with $x \geq 0$. Suppose that $\mathbf{F}(x,y) = (xy + y, y^2 x)$.

(a) Is D a type I, II, or III region or none of the above?

Solution: It is a type III region, since it can be expressed as both

$$\{(x,y): 0 \le x \le 1, x^3 \le y \le x\} \quad \text{and}$$
$$\{(x,y): 0 \le y \le 1, y \le x \le \sqrt[3]{y}\}$$

(b) Orient ∂D so that D is always on the left. Calculate $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ directly.

Solution: Parameterize the graph of $y = x$ as $(1-t, 1-t)$ and the graph of $y = x^3$ as (t, t^3) both with $0 \le t \le 1$. Notice that this gives ∂D_1 the "correct" orientation for Green's theorem.. Let C_1 and C_2 be the pieces of ∂D_1 corresponding to $y = x^3$ and $y = x$ respectively. Then:

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} &= \\
\int_0^1 \begin{pmatrix} (1-t)^2 + (1-t) \\ (1-t)^3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} t^4 + t^3 \\ t^7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} dt &= \\
\int_0^1 -(1-t)^2 - (1-t) - (1-t)^3 + (t^4 + t^3) + 3t^9 \, dt &= \\
(1-t)^3/3 + (1-t)^2/2 + (1-t)^4/4 + t^5/5 + t^4/4 + 3t^{10}/10 \Big|_0^1 &= \\
1/5 + 1/4 + 3/10 - 1/3 - 1/2 - 1/4 &= \\
-1/3 &
\end{aligned}$$

(c) Calculate $\iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA$ directly.

Solution:

$$\begin{aligned}
\int_0^1 \int_{x^3}^x \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA &= \\
\int_0^1 \int_{x^3}^x y^2 - x - 1 \, dy \, dx &= \\
\int_0^1 x^3/3 - x^2 - x - x^9/3 + x^4 + x^3 \, dx &= \\
1/12 - 1/3 - 1/2 - 1/30 + 1/5 + 1/4 &= \\
-1/3 &
\end{aligned}$$

(d) What is the relevance of Green's theorem to the preceding problems?

Solution: Since $\mathbf{F}$ is defined on D and since ∂D is piecewise C^1, Green's theorem asserts the previous two calculations should be equal. Which they are.

(e) Is the vector field $\mathbf{F}$ conservative?

Solution: No. If it were conservative the integral $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ would be 0. (There are other possible reasons.)

(7) What is the flux of the vector field $\mathbf{F}(x,y) = (-y^2x, x^2y)$ across the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

Solution: Let $\mathbf{x}(t) = (2\cos t, 2\sin t)$ for $0 \leq t \leq 2\pi$. The unit normal pointing outside the region bounded by the circle is $\mathbf{n}(t) = (\cos t, \sin(t))$. Consequently, the flux is

$$\int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \begin{pmatrix} -8\sin^2 t \cos t \\ 8\cos^2 t \sin t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} (2)\, dt.$$

This is equal to:

$$2\int_0^{2\pi} -8\cos^2 t \sin^2 t + 8\sin^2 t \cos^2 t\, dt = 0.$$

(8) What is the circulation of the vector field $\mathbf{F}(x,y) = (-y^2x, x^2y)$ around the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

Solution: We use the same notation as in the previous problem. The circulation of the vector field is:

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \begin{pmatrix} -8\sin^2 t \cos t \\ 8\cos^2 t \sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt.$$

This is equal to:

$$\int_0^{2\pi} 16\sin^3 t \cos t + 16\cos^3 t \sin t\, dt.$$

(9) Recall that if two particles, each with charge +1 are at points $\mathbf{p}$ and $\mathbf{q}$ respectively, the electric force exerted by the particle at $\mathbf{p}$ on the particle at $\mathbf{q}$ is $\frac{1}{||\mathbf{p}-\mathbf{q}||^3}(\mathbf{q}-\mathbf{p})$.

A wire C is bent into the shape of a circle of radius 1 centered at the origin in $\mathbb{R}^2$. It is given a charge of $+1$ and so generates an electric field $\mathbf{F}$. How much work is done in moving a particle with charge $+1$ from $(1/2, 0)$ to $(0,0)$? Does it matter what path is taken? Why not? (You may leave your answer in integral form.)

Solution: The scalar field $g(\mathbf{y}) = \frac{-1}{||\mathbf{x}-\mathbf{y}||}$ is a potential function for $\mathbf{G}(\mathbf{y}) = \frac{1}{||\mathbf{x}-\mathbf{y}||^3}(\mathbf{y}-\mathbf{x})$.

By the principle of superposition, we can obtain a potential function for **F** by calculation:

$$f(a,b) = \int_C \frac{-1}{\sqrt{(x-a)^2+(y-b)^2}}\,ds$$

since $\frac{-1}{\sqrt{a^2+b^2}}$ is a potential function for the electric field generated by a single particle at the origin. Choosing the usual parameterization for C and letting $b=0$, we obtain:

$$f(a,0) = -\int_0^{2\pi} \frac{1}{\sqrt{1-2a\cos t+a^2}}\,dt.$$

Since we have a potential function we can simply evaluate f on the endpoints of the path (the path not mattering the slightest) and subtract in order to find the work. So for (a) we obtain:

$$f(0,0) - f(1/2,0) = -(2\pi - \int_0^{2\pi} \frac{1}{\sqrt{1-\cos t+1/4}}\,dt)$$

(10) Suppose that C_1 and C_2 are C^1 paths bounding a compact region A in $\mathbb{R}^2$. Suppose that **F** is a C^1 vector field defined on A such that the scalar curl of **F** is a constant 9. State and explain the relationship between $\int_{C_1} \mathbf{F}\cdot d\mathbf{s}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{s}$ if both are oriented counter-clockwise.

Solution: Since **F** is C^1 on the compact region A, we may apply Green's theorem to find:

$$9\text{Area}(A) = \iint\limits_D \text{scalar curl}\mathbf{F}\,dA = \int\limits_{\partial A} \mathbf{F}\cdot d\mathbf{s}.$$

One of C_1 or C_2 is oriented so that A is on the right. The other one is oriented so that A is on the left. Hence,

$$\int\limits_{\partial A} \mathbf{F}\cdot d\mathbf{s} = \pm\int_{C_1} \mathbf{F}\cdot d\mathbf{s} \mp \int_{C_2} \mathbf{F}\cdot d\mathbf{s}.$$

Since this equals $9\text{Area}(A)$, the two line integrals differ by $9\text{Area}(A)$.

(11) Suppose that C_1 and C_2 are C^1 simple closed curves bounding a region A in $\mathbb{R}^2$ so that A is on the left of both C_1 and C_2. Suppose that **F** is a C^1 vector field defined on A such that the divergence of **F** is a constant 9. State and explain the relationship between the flux of **F** through C_1 and the flux of **F** through C_2.

Solution: Since **F** is C^1 on the compact region A, we can use the planar divergence theorem. The flux through the boundary of A is equal to $\pm\int_{C_1}\mathbf{F}\cdot d\mathbf{s}\mp\int_{C_2}\mathbf{F}\cdot d\mathbf{s}$. The planar divergence theorem says this is equal to $\iint\limits_D \operatorname{div}\mathbf{F}\,dA = 9\text{Area}(A)$. Hence, the flux through C_1 and the flux through C_2 differ by 9 times the area of A.

(12) (Challenge!) Suppose that D is the region obtained from $\mathbb{R}^2$ by removing 2 points $\mathbf{p}_1$ and $\mathbf{p_2}$. Suppose that $\mathbf{F}$ is a C^1 vector field defined on D with curl constantly zero.

(a) Are there simple closed curves $C_1, C_2, \ldots$ in D such that the sequence $\left(\int_{C_i}\mathbf{F}\cdot d\mathbf{s}\right)$ diverges to infinity?

Solution: By Green's theorem any two disjoint curves oriented in the same direction and bounding a region A not containing either $\mathbf{p}_1$ or $\mathbf{p}_2$ must produce the same value when $\mathbf{F}$ is integrated along them. (Be sure you understand this. HW 7 had several problems on this idea.) A simple closed curve in $\mathbb{R}^2$ that does not pass through either $\mathbf{p}_1$ or $\mathbf{p}_2$ has either 0, 1, or 2 of the points in the compact region with boundary the curve. Thus, if C is a simple closed curve the integral $\int_C\mathbf{F}\cdot d\mathbf{s}$ can take one of only four possible values. (Do you see why?)

(b) What if we only require $C_1, C_2, \ldots$ to be closed (and not simple)?

Solution: Suppose that $\mathbf{p}_1 = (0,0)$ and that $\mathbf{p}_2 = (8,0)$ (or anything else far away from $\mathbf{p}_1$. Define $C_n(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi n]$. Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix}-y\\ x\end{pmatrix}$. A calculation shows that $\int_{C_n}\mathbf{F}\cdot d\mathbf{s} = 2\pi n$. This sequence diverges to infinity as $n \to \infty$.

(c) What if $C_1, C_2, \ldots$ are simple closed curves, but the scalar curl of $\mathbf{F}$ is always 1 (instead of 0)?

Solution: Let M be the maximum of $||\mathbf{p}_1||$ and $||\mathbf{p}_2||$. Let C_i be a circle of radius $M+i$ oriented counterclockwise. The area between C_i and C_{i+1} is

$$\pi(M+i+1)^2 - \pi(M+i)^2 = \pi(2M+2i+1).$$

By problem (10), we have

$$\int_{C_{i+1}}\mathbf{F}\cdot d\mathbf{s} - \int_{C_i}\mathbf{F}\cdot d\mathbf{s} = \pi(2M+2i+1)$$

Thus, the sequence $\left(\int_{C_i}\mathbf{F}\cdot d\mathbf{s}\right)$ diverges to infinity.

(13) Is the vector field $\mathbf{F}(x,y,z) = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ conservative on its domain? Explain.

Solution: An easy calculation shows that the curl of $\mathbf{F}$ is the zero vector. Since $\mathbb{R}^3 - \mathbf{0}$ is simply connected, by Poincaré's theorem, $\mathbf{F}$ is conservative.

(14) Is the vector field $\mathbf{F}(x,y,z) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ conservative on its domain?

Solution: No. It is easy to see that $\mathbf{F}$ has a closed flow line tracing out the unit circle in the xy plane.

(15) Find a single variable integral representing the area enclosed by the path $\phi(t) = (2\cos(2t), 3\sin(3t))$ for $-\pi/3 \le t \le \pi/3$.

Solution: We note that the orientation of the path ϕ has the bounded region D always on the left. Hence by Green's theorem and the fact that $\operatorname{curl}\begin{pmatrix} 0 \\ x \end{pmatrix} = 1$:

$$\begin{aligned} \iint_D 1\,dA &= \int_{-\pi/3}^{\pi/3} \begin{pmatrix} 0 \\ 2\cos 2t \end{pmatrix} \cdot \begin{pmatrix} -4\sin 2t \\ 9\cos 3t \end{pmatrix} dt \\ &= \int_{-\pi/3}^{\pi/3} 18\cos(2t)\cos(3t)\,dt. \end{aligned}$$

(16) Let $\sigma\colon [1,2] \to \mathbb{R}^2$ be the path $\sigma(t) = (e^{t-1}, \sin(\pi/t))$. Let $\mathbf{F}(x,y) = (2x\cos y, -x^2\sin y)$. Compute $\int_\sigma \mathbf{F}\cdot d\mathbf{s}$.

Hint: Show and then use the fact that the vector field is conservative.

Solution: It is easy to see that $f(x,y) = x^2\cos y$ is a potential function for $\mathbf{F}$. The requested integral is then equal to $f(\sigma(2)) - f(\sigma(1))$. You could also choose a nicer path joining the endpoints of σ and integrate over that instead.

(17) Find a parameterization of the surface formed by the graph of $z = x^2 - y^2$ with (x,y) in the triangle in the xy-plane formed by the x-axis, the y-axis, and the line $y = -x+1$.

Solution: How about:

$$\mathbf{X}(s,t) = \begin{pmatrix} s \\ t \\ s^2 - t^2 \end{pmatrix}$$

with $0 \le s \le 1$ and $0 \le t \le -s+1$?

(18) Is the surface in the previous problem a smooth surface? If no, at what points is it not smooth?

Solution: The answer depends (somewhat) on your parameterization. The answer here is based on the parameterization above.

You can calculate that

$$\begin{aligned} \mathbf{T}_s &= (1,0,2s) \\ \mathbf{T}_t &= (0,1,-2t) \\ \mathbf{N} &= (-2s,2t,1) \end{aligned}$$

Since $\mathbf{N}$ is never $\mathbf{0}$, and since $\mathbf{X}$ is obviously C^1, $\mathbf{X}$ is a smooth surface.

(19) Find a parameterization of the surface formed by rotating the curve $\begin{pmatrix} \cos t + 5 \\ 2\sin t \end{pmatrix}$ with $0 \le t \le 2\pi$ around the y-axis.

Solution: How about

$$\mathbf{X}(s,t) = \begin{pmatrix} \cos s(\cos t + 5) \\ 2\sin t \\ \sin s(\cos t + 5) \end{pmatrix}$$

for $0 \le t \le 2\pi$ and $0 \le s \le 2\pi$?

(20) Consider the surface

$$\mathbf{X}(s,t) = \begin{pmatrix} 2\sin 3t + t \\ \cos 2s \\ t^2 + s^2 \end{pmatrix}, \quad 0 \le t \le \pi/4, \quad 0 \le s \le \pi$$

Find the tangent and normal vectors to $\mathbf{X}$ at the point $(\pi/6, \pi/6)$. Is the surface smooth?

Solution:

We have

$$\begin{aligned} \mathbf{T}_s &= (0,-2\sin 2s, 2s) \\ \mathbf{T}_t &= (6\cos(3t)+1,0,2t) \\ \mathbf{N} &= (-4t\sin 2s, 2s(6\cos 3t+1), 2\sin 2s(6\cos 3t+1)) \end{aligned}$$

Plug $(\pi/6, \pi/6)$ into the above equations to get:

$$\begin{aligned} \mathbf{T}_s &= (0, -\sqrt{3}, \pi/3) \\ \mathbf{T}_t &= (1, 0, \pi/3) \\ \mathbf{N} &= (-\pi\sqrt{3}/3, \pi/3, \sqrt{3}) \end{aligned}$$

Since $\mathbf{N}(\pi/6, \pi/6) \neq \mathbf{0}$, the surface is smooth at that point.

(21) Let S be the disc of radius 1 centered at $(1,0,0)$ in $\mathbb{R}^3$ which is parallel to the yz-plane. Orient S with normal vector pointing in the direction of the postive x-axis. Use the definition of surface integral to calculate the flux of $\mathbf{F}(x,y,z) = (-xy, yz, xz)$ through S.

Solution: Parameterize S as:

$$\mathbf{X}(s,t) = \begin{pmatrix} 1 \\ s \\ t \end{pmatrix}$$

with (s,t) in the region D defined by $0 \leq s^2 + t^2 \leq 1$. It is easy to calculate $\mathbf{N} = (1,0,0)$. Then,

$$\mathbf{F} \cdot \mathbf{N}(x,y,z) = -xy.$$

Thus, by the definition of surface integral, the flux of $\mathbf{F}$ through S is

$$\iint_D \mathbf{F} \cdot \mathbf{N}(\mathbf{X}(s,t))\, dA = \iint_D -s\, ds\, dt.$$

Change to polar coordinates by setting $s = r\cos\theta$ and $t = r\sin\theta$. Then the integral above is equal to (by the change of coordinates theorem):

$$\int_0^1 \int_0^{2\pi} -r^2 \cos\theta\, d\theta dr$$

Since $\int_0^{2\pi} \cos\theta d\theta = 0$, the flux equals 0.

(22) Use the same surface S and $\mathbf{F}$ as in the previous problem, but now use Stoke's theorem to calculate the flux of the curl from the previous problem.

Solution: By Stoke's theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\, d\mathbf{s}.$$

Parameterize ∂S as:

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}$$

with $0 \leq t \leq 2\pi$.

Notice that $\mathbf{x}$ gives ∂S the orientation induced by the orientation on S. Then,

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x})(t) \cdot \mathbf{x}'(t)\, dt.$$

Calculations show that this equals

$$\begin{aligned} \textstyle\int_0^{2\pi} -\cos t \sin^2 t + \sin t \cos t \, dt &= \textstyle\int_0^{2\pi} -\cos t \sin^2 t \, dt + \int_0^{2\pi} \sin t \cos t \, dt \\ &= 0. \end{aligned}$$