MA 302: Practice Exam 2 **Name:**________________________

(1) Give an example of a vector field $\mathbf{F}$ having $\operatorname{curl}\mathbf{F} = \mathbf{0}$, but where $\mathbf{F}$ is not a gradient field.

(2) Know the formal definitions of the following terms or the complete precise statements of the following theorems:

(a) Green's theorem

(b) planar divergence theorem

(c) Stokes' theorem

(d) Poincaré's theorem

(e) parameterized surface

(f) orientable surface

(g) one-sided surface

(3) Give an example of a one-sided surface in $\mathbb{R}^3$.

(4) Give an example of an orientable surface in $\mathbb{R}^3$.

(5) Be able to do the following:

(a) Suppose that $D \subset \mathbb{R}^2$ is the union of two "nice" regions D_1 and D_2 along an edge C in their boundaries. Suppose that $\mathbf{F}$ is a C^1 vector field defined on the union $D = D_1 \cup D_2$. Prove that $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{s}$.

(b) Give an outline of the proof of Green's theorem.

(c) Suppose that $X \subset \mathbb{R}^2$ is a simply connected open subset and that $\mathbf{F}\colon X \to \mathbb{R}^2$ has $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Prove that if C is a simple closed curve in X then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$. Use this to prove that $\mathbf{F}$ is conservative.

(6) Let $D \subset \mathbb{R}^2$ be the region bounded by the graphs of the equations $y = x^3$ and $y = x$ and with $x \geq 0$. Suppose that $\mathbf{F}(x,y) = (xy + y, y^2 x)$.

(a) Is D a type I, II, or III region or none of the above?

(b) Orient ∂D so that D is always on the left. Calculate $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ directly.

(c) Calculate $\iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$ directly.

(d) What is the relevance of Green's theorem to the preceding problems?

(e) Is the vector field $\mathbf{F}$ conservative?

(7) What is the flux of the vector field $\mathbf{F}(x,y) = (-y^2x, x^2y)$ across the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

(8) What is the circulation of the vector field $\mathbf{F}(x,y) = (-y^2x, x^2y)$ around the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

(9) Recall that if two particles, each with charge +1 are at points $\mathbf{p}$ and $\mathbf{q}$ respectively, the electric force exerted by the particle at $\mathbf{p}$ on the particle at $\mathbf{q}$ is $\frac{1}{||\mathbf{p}-\mathbf{q}||^3}(\mathbf{q}-\mathbf{p})$.

A wire C is bent into the shape of a circle of radius 1 centered at the origin in $\mathbb{R}^2$. It is given a charge of $+1$ and so generates an electric field $\mathbf{F}$. How much work is done in moving a particle with charge $+1$ from $(1/2, 0)$ to $(0,0)$? Does it matter what path is taken? Why not? (You may leave your answer in integral form.)

(10) Suppose that C_1 and C_2 are C^1 paths bounding a compact region A in $\mathbb{R}^2$. Suppose that $\mathbf{F}$ is a C^1 vector field defined on A such that the scalar curl of $\mathbf{F}$ is a constant 9. State and explain the relationship between $\int_{C_1} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ if both are oriented counter-clockwise.

(11) Suppose that C_1 and C_2 are C^1 simple closed curves bounding a region A in $\mathbb{R}^2$. Assume that both C_1 and C_2 are oriented counter-clockwise. Suppose that $\mathbf{F}$ is a C^1 vector field defined on A such that the divergence of $\mathbf{F}$ is a constant 9. State and explain the relationship between the flux of $\mathbf{F}$ through C_1 and the flux of $\mathbf{F}$ through C_2.

(12) (Challenge!) Suppose that D is the region obtained from $\mathbb{R}^2$ by removing 2 points $\mathbf{p}_1$ and $\mathbf{p_2}$. Suppose that $\mathbf{F}$ is a C^1 vector field defined on D with curl constantly zero.

(a) Are there simple closed curves $C_1, C_2, \ldots$ in D such that the sequence $\left(\int_{C_i} \mathbf{F} \cdot d\mathbf{s}\right)$ diverges to infinity?

(b) What if we only require $C_1, C_2, \ldots$ to be closed (and not simple)?

(c) What if $C_1, C_2, \ldots$ are simple closed curves, but the scalar curl of $\mathbf{F}$ is always 1 (instead of 0)?

(13) Is the vector field $\mathbf{F}(x,y,z) = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ conservative on its domain? Explain.

(14) Is the vector field $\mathbf{F}(x,y,z) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ conservative on its domain?

(15) Find a single variable integral representing the area enclosed by the path $\phi(t) = (2\cos(2t), 3\sin(3t))$ for $-\pi/3 \leq t \leq \pi/3$.

(16) Let $\sigma\colon [1,2] \to \mathbb{R}^2$ be the path $\sigma(t) = (e^{t-1}, \sin(\pi/t))$. Let $\mathbf{F}(x,y) = (2x\cos y, -x^2\sin y)$. Compute $\int_\sigma \mathbf{F}\cdot d\mathbf{s}$.

Hint: Show and then use the fact that the vector field is conservative.

(17) Find a parameterization of the surface formed by the graph of $z = x^2 - y^2$ with (x,y) in the triangle in the xy-plane formed by the x-axis, the y-axis, and the line $y = -x+1$.

(18) Is the surface in the previous problem a smooth surface? If no, at what points is it not smooth?

(19) Find a parameterization of the surface formed by rotating the curve $\begin{pmatrix} \cos t + 5 \\ 2\sin t \end{pmatrix}$ with $0 \leq t \leq 2\pi$ around the y-axis.

(20) Consider the surface

$$\mathbf{X}(s,t) = \begin{pmatrix} 2\sin 3t + t \\ \cos 2s \\ t^2 + s^2 \end{pmatrix}, \quad 0 \leq t \leq \pi/4, \quad 0 \leq s \leq \pi$$

Find the tangent and normal vectors to $\mathbf{X}$ at the point $(\pi/6, \pi/6)$. Is the surface smooth?

(21) Let S be the disc of radius 1 centered at $(1,0,0)$ in $\mathbb{R}^3$ which is parallel to the yz-plane. Orient S with normal vector pointing in the direction of the postive x-axis. Use the definition of surface integral to calculate the flux of $\mathbf{F}(x,y,z) = (-xy, yz, xz)$ through S.

(22) Use the same surface S and $\mathbf{F}$ as in the previous problem, but now use Stoke's theorem to calculate the flux of the curl from the previous problem.