

- (1) Let  $F(x, y) = (x^2y, y^2x, 3x - 2yx)$ . Find the derivative of  $F$ .

**Solution:**

$$DF(x, y) = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \\ 3 - 2y & -2x \end{pmatrix}$$

- (2) Let  $F(x, y) = (x - y, x + y)$  and let  $G(x, y) = (x \cos y, x \sin y)$ . Find the derivative of  $F \circ G$  using the chain rule.

**Solution:**

$$DF(x, y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$DG(x, y) = \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix}$$

$$\begin{aligned} D(F \circ G)(x, y) &= DF(G(x, y))DG(x, y) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix} \\ &= \begin{pmatrix} \cos y - \sin y & -x \sin y - x \cos y \\ \cos y + \sin y & -x \sin y + x \cos y \end{pmatrix} \end{aligned}$$

- (3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time  $t$  seconds the center of the circle is at the point  $(t, \sin t)$ . Let  $P$  be the point on the circle which is at  $(0, 1)$  at time  $t = 0$ . If the circle makes 3 revolutions per second, what is the path  $\mathbf{x}(t)$  taken by the point  $P$ ?

**Solution:** The rotation of the  $P$  relative to the center of the circle (that is, in  $T_{\mathbf{c}(t)}$ ) can be described by the path  $(\cos(6\pi t + \pi/2), \sin(6\pi t + \pi/2))$ . Thus,  $\mathbf{x}(t) = (\cos(6\pi t + \pi/2) + t, \sin(6\pi t + \pi/2) + \sin t)$ .

- (4) A rotating circle of radius 1 follows a helical path in  $\mathbb{R}^3$  so that at time  $t$  the center of the circle is at  $(\sin t, \cos t, t)$ . At each time  $t$ , the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let

$P$  be the point on the circle which is at  $(1, 0)$  at time  $t = 0$ . The circle completes one rotation every  $2\pi$  seconds. Find a formula  $\mathbf{x}(t)$  for the path taken by the point  $P$ .

**Solution:** Relative to the center of the circle (that is, in  $T_{\mathbf{c}(t)}$ ) the point  $P$  follows the path  $\cos t\mathbf{T} + \sin t\mathbf{N}$  where  $\mathbf{T}$  and  $\mathbf{N}$  are the unit tangent and unit normal vectors to  $\mathbf{c}(t) = (\sin t, \cos t, t)$  respectively. Those formulae are

$$\begin{aligned}\mathbf{T}(t) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} \\ \mathbf{N}(t) &= \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{c}(t) &= \cos t\mathbf{T} + \sin t\mathbf{N} + \mathbf{c}(t) \\ &= \frac{\cos t}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}.\end{aligned}$$

(5) Explain what it means for curvature to be an intrinsic quantity.

**Solution:** The curvature of a path  $\mathbf{x}(t)$  at  $t_0$ , depends only on the curve itself at  $t_0$ , not on the parameterization  $\mathbf{x}$ .

(6) Prove that the curvature at any point of a circle of radius  $r$  is  $1/r$ .

**Solution:** A circle of radius  $r$  can be parameterized as  $\mathbf{x}(t) = (r \cos t, r \sin t)$  for  $0 \leq t \leq 2\pi$ . We have:

$$\begin{aligned}\mathbf{x}'(t) &= (-r \sin t, r \cos t) \\ \|\mathbf{x}'(t)\| &= r \\ \mathbf{T} &= (-\sin t, \cos t) \\ \mathbf{T}' &= (-\cos t, -\sin t) \\ \|\mathbf{T}'\| &= 1 \\ \kappa(t) &= \|\mathbf{T}'\|/\|\mathbf{x}'\| \\ &= 1/r.\end{aligned}$$

(7) Let  $\mathbf{x}(t) = (\cos t, \sin t, t)$  for  $1 \leq t \leq 2$ . Find  $\mathbf{T}$ ,  $\mathbf{N}$ , and  $\mathbf{B}$  (that is, the moving frame) for  $\mathbf{x}$  as well as  $\kappa$  and  $\tau$ .

**Solution:** We have:

$$\begin{aligned}
 \mathbf{x}'(t) &= (-\sin t, \cos t, t) \\
 \|\mathbf{x}'(t)\| &= \sqrt{2} \\
 \mathbf{T}(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\
 \mathbf{T}'(t) &= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
 \mathbf{N}(t) &= (-\cos t, -\sin t, 0) \\
 \mathbf{B}(t) &= \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1) \\
 \kappa(t) &= 1/2 \\
 \mathbf{B}'(t) &= \frac{1}{\sqrt{2}}(\cos t, \sin t, 0) \\
 \frac{\mathbf{B}'(t)}{\|\mathbf{x}'(t)\|} &= \frac{1}{2}(\cos t, \sin t, 0) \\
 \tau(t) &= 1/2.
 \end{aligned}$$

- (8) Suppose that  $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^n$  is a  $C^1$  path such that for all  $t$ ,  $\|\mathbf{x}(t)\| = 5$ . Prove that at each  $t$ ,  $\mathbf{x}(t)$  and  $\mathbf{x}'(t)$  are perpendicular.

**Solution:** Since  $5 = \mathbf{x}(t) \cdot \mathbf{x}(t)$ , by the product rule, we have  $0 = 2\mathbf{x} \cdot \mathbf{x}'$ , implying that  $\mathbf{x}$  and  $\mathbf{x}'$  are perpendicular.

- (9) A particle is following the path  $\mathbf{x}(t) = (t, t^2, t^3)$  for  $1 \leq t \leq 5$ . Find an integral representing the distance travelled by the particle after  $t$  seconds.

**Solution:** The distance travelled after  $t$  seconds is

$$\begin{aligned}
 s(t) &= \int_1^t \|\mathbf{x}'(\tau)\| d\tau \\
 &= \int_1^t \sqrt{1 + 4\tau^2 + 9\tau^4} d\tau.
 \end{aligned}$$

- (10) Let  $\mathbf{x}(t) = (t^2, 3t^2)$  for  $t \geq 1$ . Reparameterize  $\mathbf{x}$  by arc length.

**Solution:** We compute,

$$s(t) = \int_1^t \sqrt{4\tau^2 + 36\tau^2} d\tau = \int_1^t 2\tau\sqrt{10} d\tau = \sqrt{10}(t^2 - 1).$$

Then,

$$s^{-1}(t) = \sqrt{t/\sqrt{10} + 1}$$

Consequently,

$$\mathbf{y}(t) = \mathbf{x} \circ s^{-1}(t) = (t/\sqrt{10} + 1, 3t/\sqrt{10} + 3)$$

is the reparameterization of  $\mathbf{x}$  by arclength.

- (11) Suppose that  $\mathbf{x}(t)$  is a path in  $\mathbb{R}^n$  such that  $\mathbf{x}(0) = \mathbf{a}$  and  $\mathbf{x}(1) = \mathbf{b}$  (that is,  $\mathbf{x}$  is a path joining  $\mathbf{a}$  to  $\mathbf{b}$ .) Find a path which has the same image as  $\mathbf{x}$  but which joins  $\mathbf{b}$  to  $\mathbf{a}$ .

**Solution:**  $\mathbf{y}: [-1, 0] \rightarrow \mathbb{R}^n$  defined by  $\mathbf{y}(t) = \mathbf{x}(-t)$  will do the trick since  $\mathbf{y}(0) = \mathbf{a}$  and  $\mathbf{y}(-1) = \mathbf{b}$ .

- (12) Let  $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^n$  be a path with  $\mathbf{x}'(t) \neq \mathbf{0}$  for all  $t$ . Let  $\mathbf{y} = \mathbf{x} \circ \phi$  be an orientation reversing reparameterization of  $\mathbf{x}$ . Suppose that  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  is integrable. Prove that  $\int_{\mathbf{y}} f ds = \int_{\mathbf{x}} f ds$ .

**Solution:** Since  $\phi$  is orientation reversing,  $|\phi'(t)| = -\phi'(t)$ . Hence,  $\|\mathbf{y}'(t)\| = -\|\mathbf{x}'(\phi(t))\|\phi'(t)$ . Thus,

$$\int_{\mathbf{y}} f ds = - \int_c^d f(\mathbf{x}(\phi(t))) \|\mathbf{x}'(\phi(t))\| \phi'(t) dt.$$

Substitute  $u = \phi(t)$  and  $du = \phi'(t)dt$  to get:

$$\int_{\mathbf{y}} f ds = - \int_b^a f(\mathbf{x}(u)) \|\mathbf{x}'(u)\| du.$$

Reversing the limits of integration eliminates the negative sign and so the result follows.

- (13) Let  $\mathbf{x}(t) = (t \cos t, t \sin t)$  for  $0 \leq t \leq 2\pi$ . Let  $f(x, y) = y \cos x$ . Let  $F(x, y) = (-y, x)$ . Find one-variable integrals representing  $\int_{\mathbf{x}} f ds$  and  $\int_{\mathbf{x}} F \cdot ds$ .

**Solution:** Notice that

$$\begin{aligned} \mathbf{x}(t) &= (t \cos t, t \sin t) \\ \mathbf{x}'(t) &= (\cos t - t \sin t, t \cos t + \sin t) \\ \|\mathbf{x}'(t)\| &= \sqrt{(\cos t - t \sin t)^2 + (t \cos t + \sin t)^2} \end{aligned}$$

Thus,

$$\int_{\mathbf{x}} f ds = \int_0^{2\pi} t \sin t \cos(t \cos t) \sqrt{(\cos t - t \sin t)^2 + (t \cos t + \sin t)^2} dt.$$

And,

$$\begin{aligned} \int_{\mathbf{x}} F \cdot ds &= \int_0^{2\pi} \begin{pmatrix} -t \sin t \\ t \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t - t \sin t \\ \sin t + t \cos t \end{pmatrix} dt \\ &= \int_0^{2\pi} t^2 dt \\ &= 8\pi^3/3. \end{aligned}$$

- (14) Let  $F(x, y) = (ax+by, cx+dy)$  be a transformation of space, where  $a, b, c, d$  are constants such that  $ad - bc \neq 0$ . Suppose that an object is moving in a circle  $\mathbf{x}(t) = (\cos t, \sin t)$ . Let  $\mathbf{y}(t) = F(\mathbf{x}(t))$ . If all forces stop acting on an object following the path  $\mathbf{y}$  at time  $t = \pi$ , where will the object be 3 seconds later?

**Solution:** We find that  $\mathbf{y}(\pi) = F(\mathbf{x}(\pi)) = (-a, -c)$ . By the chain rule,

$$\mathbf{y}'(t) = DF D\mathbf{x} = (-a \sin t + b \cos t, -c \sin t + d \cos t).$$

Thus the object will follow the path

$$l(t) = t(-b, -d) + (-a, -c)$$

at time  $t + \pi$  seconds. Hence, at time  $t = \pi + 3$ , the particle is at  $l(3) = (-3b - a, -3d - c)$ .

(15) Let  $\mathbf{F}(x, y) = (x, -2y)$ .

- Sketch a portion of the vector field  $F$ .
- Sketch a flow line for the vector field starting at  $(1, 1)$ .
- Find a parameterization for the flow line starting at  $(1, 1)$ .

**Solution:** Let  $\phi(t) = (x(t), y(t))$  be the flow line. Then we are looking for  $x$  and  $y$  so that:

$$\begin{aligned} x'(t) &= x(t) \\ y'(t) &= -2y(t) \\ x(0) &= 1 \\ y(0) &= 1 \end{aligned}$$

Using  $x(t) = e^t$  and  $y(t) = e^{-2t}$  does the trick.

- The vector field  $F$  is a gradient field. Find the potential function.

**Solution:**  $f(x, y) = x^2/2 - y^2$ .

(16) Let  $F(x, y) = (2xy, x^2 + 1)$ . Find a potential function for  $F$ .

**Solution:**  $f(x, y) = x^2y + y$ .

(17) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that  $\mathbf{F}(x, y) = (-y, x)$  is not a gradient field.

**Solution:** See course notes.

(18) Suppose that  $\mathbf{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a gradient field with potential function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ . Let  $\phi$  be a flow line for  $\mathbf{F}$  and let  $L$  be an equipotential line (i.e. contour line) for  $f$ . Prove that if  $\phi$  and  $L$  intersect, they do so at right angles. (That is, they are perpendicular.)

**Solution:** See course notes.

(19) Let  $f(x, y) = ye^x$ . Find the gradient of  $f$ .

**Solution:**  $\nabla f(x, y) = (ye^x, e^x)$ .

(20) Let  $F(x, y, 0) = (ye^x, xe^{y^2}, 0)$ . Find the divergence of  $F$ .

**Solution:**  $\operatorname{div} f(x, y) = ye^x + 2yxe^{y^2}$

(21) Let  $F(x, y, z) = (xyz, xe^y \ln(z), x^2 + y^2 + z^2)$ . Find the curl of  $F$ .

**Solution:**

$$\begin{pmatrix} 2y - xe^y/z \\ xy - 2x \\ e^y \ln z - xz \end{pmatrix}$$

(22) Find the curl of your answer to problem 16.

**Solution:0**

(23) Find the divergence of your answer to problem 18.

**Solution: 0.**