MA 302: HW 8 additional problems

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A: Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. This vector field is defined on $\mathbb{R}^2 - \{0\}$. We know the following facts (all of which are 100% guaranteed!):

(1) $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

(2) $\mathbf{F}$ is not a gradient field on $\mathbb{R}^2 - \{0\}$.

(3) The disc D of radius 4 centered at $(5,0)$ is simply connected and so $\mathbf{F}$ is a gradient field on D.

Explain how it is possible for $\mathbf{F}$ to be a gradient field on D but not on all of $\mathbb{R}^2 - \{\mathbf{0}\}$.

For a slightly stranger example, consider the sets

$$X_- = \{(x,y) : \text{ if } y = 0 \text{ then } x < 0\}$$

and

$$X_+ = \{(x,y) : \text{ if } y = 0 \text{ then } x > 0\}.$$

Both sets X_- and X_+ are simply connected and they have all points not on the x-axis in common. Let $\mathbf{a} = (1,1)$. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, there is a potential function f_- for $\mathbf{F}$ on X_- where $f_-(\mathbf{x}) = \int_\phi \mathbf{F}\cdot d\mathbf{s}$ with ϕ any path in X_- joining $\mathbf{a}$ to $\mathbf{x}$. Similarly, there is a potential function f_+ for $\mathbf{F}$ on X_+ where $f_+(\mathbf{x}) = \int_\psi \mathbf{F}\cdot d\mathbf{s}$ with ψ any path in X_+ joining $\mathbf{a}$ to $\mathbf{x}$.

Bonus Question: Let $\mathbf{x}$ be a point with negative y-coordinate. Without doing any significant calculation, tell me what $f_-(\mathbf{x}) - f_+(\mathbf{x})$ is.

Problem B: Suppose that $D \subset \mathbb{R}^2$ is an open region and that $\mathbf{H}$ is a C^1 vector field on D having the property that for any closed curve C, the integral $\int_C \mathbf{H}\cdot d\mathbf{s}$ is zero. Recall that this implies that $\mathbf{H}$ is a gradient field. That is, there exists a scalar field f on D such that $\mathbf{H} = \nabla f$. Use this fact to prove the following:

Suppose that $\mathbf{F}$ and $\mathbf{G}$ are C^1 vector fields on D with the property that for every oriented closed curve C in D,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{G} \cdot d\mathbf{s}.$$

Then there exists a scalar field $f\colon D \to \mathbb{R}^2$ such that $\mathbf{F} = \mathbf{G} + \nabla f$.

Problem C: (Extra-Credit) Use Problem B and the additional problems from HW 7 to prove the following:

Let $D = \mathbb{R}^2 - \{\mathbf{0}\}$ and let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Suppose that $\mathbf{G}$ is a C^1 vector field on D such that $\operatorname{curl} \mathbf{G} = \mathbf{0}$.

Let C_1 be the circle of radius 1 centered at $\mathbf{0}$. Define:

$$k = \frac{1}{2\pi} \int_{C_1} \mathbf{G} \cdot d\mathbf{s}.$$

Prove that there exists a scalar field f on D such that:

$$\mathbf{G} = k\mathbf{F} + \nabla f.$$

This shows that every vector field on D with zero curl is a scalar multiple of the vector field $\mathbf{F}$ plus a gradient field.

For those who have had linear algebra: here is a way of rephrasing this result. Let V be the real vector space consisting of C^1 vector fields on D with $\mathbf{0}$ curl. Let W be the real vector space consisting of C^1 gradient fields on D. Then the quotient vector space V/W is 1-dimensional. This vector space has the name "the first cohomology group of D" and is denoted $H^1(D)$.

Problem D: (even more extra-credit) Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be n distinct points in $\mathbb{R}^2$. Let $D = \mathbb{R}^2 - \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$. Prove that $H^1(D)$ is an n-dimensional vector space.

(If this interests you, you may be interested in the classroom scene in the movie "A Beautiful Mind". You can see part of the scene here:

`http://www.haverford.edu/math/lbutler/MITclassroom.mov`

Pay attention to the problem on the board. Can you spot the difference between the problem stated in the movie and the version appearing here?

Read an article about the scene at:

`http://www.sciencemag.org/cgi/reprint/295/5556/789.pdf`)