MA 302: HW 7 additional problems

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A: Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a C^2 scalar field. Prove that $\operatorname{curl}(\nabla f) = \mathbf{0}$. (This was discussed a while ago, but never proven. You can do it by a simple, somewhat tedious calculation using the formula for curl in terms of partial derivatives.)

Problem B: Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Let D_r be closed disc of radius $r > 0$ centered at the origin.

(1) Compute $\int_{\partial D_r} \mathbf{F} \cdot d\mathbf{s}$. Does your answer depend on r? (You may simply quote your previous HW if you wish)

(2) Explain why Green's theorem cannot be used to calculate the line integral in the previous part.

(3) Recall that $\operatorname{curl}\mathbf{F}(x,y) = \mathbf{0}$.

(4) Let $\mathbf{x}\colon [0,2\pi] \to \mathbb{R}^2$ be any simple closed curve enclosing the origin. Assume that $\mathbf{x}$ is oriented counterclockwise around the origin. Use Green's theorem and part (1) to calculate $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$.

Problem C: Suppose that $\mathbf{F}$ is a vector field which is defined and is C^1 on all of $\mathbb{R}^2$ except for n points: $\mathbf{p}_1, \dots, \mathbf{p}_n$. Suppose that $\operatorname{curl}\mathbf{F}(x,y) = \mathbf{0}$ for all $(x,y) \in \mathbb{R}^2 - \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$. Let C_1 and C_2 be two disjoint simple closed curves in $\mathbb{R}^2 - \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$, both oriented counterclockwise or both oriented clockwise. C_2 bounds a closed, bounded region inside $\mathbb{R}^2$; assume that C_1 is contained in that region. Let A be the region between C_1 and C_2.

Prove that if A does not contain any of the points $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

Problem D: The point of this problem is to prove that curl is a linear operator on the vector space of C^1 vector fields. For this problem you can use either the integral definition of curl or the partial derivative definition of curl, although the integral version makes the problem much easier, because you can appeal to properties of integrals.

(1) Suppose that $\mathbf{F}$ and $\mathbf{G}$ are C^1 vector fields both defined on a given open set in $\mathbb{R}^2$. Explain why
$$\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}.$$
(2) Suppose that $\mathbf{F}$ is a C^1 vector field defined on an open set in $\mathbb{R}^2$. Let k be a constant. Explain why $\operatorname{curl}(k\mathbf{F}) = k\operatorname{curl}(\mathbf{F})$.

Problem E: Using earlier problems on this assignment you can answer this question with very little calculation.

Let $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = (1,0)$. Define:
$$\begin{aligned}
\mathbf{F_a}(x,y) &= \tfrac{1}{||(x,y)||^2}\begin{pmatrix}-y\\ x\end{pmatrix}\\
\mathbf{F_b}(\mathbf{x}) &= \mathbf{F_a}(\mathbf{x}-\mathbf{b})\\
\mathbf{F}(\mathbf{x}) &= 7\mathbf{F_a}(\mathbf{x}) + (9/\pi)\mathbf{F_b}(\mathbf{x}).
\end{aligned}$$

(1) Let C_1 be a simple closed curve enclosing $\mathbf{a}$ but not $\mathbf{b}$ and that is oriented counterclockwise. What is $\int_{C_1}\mathbf{F}\cdot d\mathbf{s}$?
(2) Let C_2 be a simple closed curve enclosing $\mathbf{b}$ but not $\mathbf{a}$ and that is oriented counterclockwise. What is $\int_{C_2}\mathbf{F}\cdot d\mathbf{s}$?
(3) If C_3 is a simple closed curve enclosing both $\mathbf{a}$ and $\mathbf{b}$ and that is oriented counterclockwise, what is $\int_{C_3}\mathbf{F}\cdot d\mathbf{s}$?

This problem suggests that the structure of the set of vector fields on an open region in $\mathbb{R}^2$ might be completely determined by the topology of simple closed curves in the region. Pursuing this idea will lead one to the concept of Poincaré duality between cohomology (the algebra of vector fields and scalar fields) and homology (the algebra of simple closed curves). This is a very useful concept in modern mathematics.