MA 302: HW 6 – Conservative Vector Fields

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A: Suppose that P is a particle with charge $+1$ at $\mathbf{x} \in \mathbb{R}^2$ and that Q is a particle with charge $+1$ at position $\mathbf{a} \in \mathbb{R}^2$. Coulomb's law states that the force (electric field) exerted by P on Q is (up to a constant of proportionality):

$$\mathbf{F}(\mathbf{a}) = \frac{1}{||\mathbf{a}-\mathbf{x}||^3}(\mathbf{a}-\mathbf{x})$$

(1) Show that $f(\mathbf{a}) = \frac{-1}{||\mathbf{x}-\mathbf{a}||}$ is a potential function for $\mathbf{F}$.
(2) Suppose that P is at $\mathbf{x} = \mathbf{0}$ and that Q travels along the path $\mathbf{a}(t) = \begin{pmatrix} r \\ t \end{pmatrix}$ for some fixed $r > 0$ and $0 \leq t \leq 5$. Compute the work done by the electric field exerted by P on Q. (Remember that the electric field is conservative!)

Problem B: Suppose that P_1, P_2, and P_3 are particles each with charge $+1$ at positions $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3} \in \mathbb{R}^2$ and that Q is a particle with charge $+1$ at position $\mathbf{a} \in \mathbb{R}^2$. The law of superposition states that the force (electric field) exerted by P_1, P_2, and P_3 on Q is (up to a constant of proportionality):

$$\mathbf{F}(\mathbf{a}) = \sum_{i=1}^{3} \frac{1}{||\mathbf{a}-\mathbf{x_i}||^3}(\mathbf{a}-\mathbf{x_i})$$

(1) Find a potential function for $\mathbf{F}$.
(2) Suppose that $\mathbf{x_1} = \mathbf{0}$, $\mathbf{x_2} = (0,1)$ and $\mathbf{x_3} = (0,2)$. Suppose that Q travels along the path $\mathbf{a}(t) = \begin{pmatrix} r \\ t \end{pmatrix}$ for some fixed $r > 0$ and $0 \leq t \leq 5$. Compute the work done by the electric field exerted by P_1, P_2, and P_3 on Q. (Remember that the electric field is conservative!)

Problem C: Suppose that $\mathscr{P}$ is a wire, each point of which has charge +1 and suppose that Q is a particle with charge $+1$ at position $\mathbf{a} \in \mathbb{R}^2$. Let $\mathbf{F}$ be the electric field generated by $\mathscr{P}$. Think of $\mathbf{F}$ as the force exerted by $\mathscr{P}$ on Q.

(1) Use superposition to find a potential function for $\mathbf{F}$.
(2) Suppose that $\mathscr{P}$ is the portion of the y-axis between $\mathbf{0}$ and $(0,5)$. Suppose that Q travels along the path $\mathbf{a}(t) = \begin{pmatrix} r \\ t \end{pmatrix}$ for some fixed $r > 0$ and $0 \leq t \leq 5$. Compute the work done by the electric field exerted by $\mathscr{P}$ on Q. (Remember that the electric field is conservative!)

Problem D: For this problem assume that $\mathbf{F}$ is an (arbitrary) C^1 vector field defined on an open region $D \subset \mathbb{R}^2$. Recall that to say that $\mathbf{F}$ has path independent line integrals means that for all points $\mathbf{a}$ and $\mathbf{b}$ in D and for any two paths $\mathbf{x}$ and $\mathbf{y}$ in D joining $\mathbf{a}$ to $\mathbf{b}$ then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$. Before attempting these problems, you may wish to review Theorem 1.5 on page 371.

(1) Assume that $\mathbf{F}$ has path independent line integrals. Using only this fact and properties of line integrals, but not anything about conservative vector fields, prove that if $\phi : [a,b] \to D$ is a closed curve (i.e. $\phi(a) = \phi(b)$) then $\int_\phi \mathbf{F} \cdot d\mathbf{s} = 0$.
(2) Assume that $\mathbf{F}$ has the property that if ϕ is a closed curve in D then $\int_\phi \mathbf{F} \cdot d\mathbf{s} = 0$. Prove (using only this fact and properties of line integrals) that $\mathbf{F}$ has path independent line integrals.

Problem E: Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$.

(1) Let $\phi_r(t) = r \begin{pmatrix} \cos(t/r^2) \\ \sin(t/r^2) \end{pmatrix}$. Show that ϕ_r is a flow line for $\mathbf{F}$ through the point $(r,0)$.
(2) Calculate $\int_{\phi_r} \mathbf{F} \cdot d\mathbf{s}$.
(3) Let $\mathbf{a} \in \mathbb{R}^2 - \{\mathbf{0}\}$ be an arbitrary point. Let $\psi_r(t) = \mathbf{a} + r \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ where r is a fixed number such that $0 < r < ||\mathbf{a}||$ and $0 \leq t \leq 2\pi$. Compute $\int_{\psi_r} \mathbf{F} \cdot d\mathbf{s}$.

(4) Using the definition of scalar curl of $\mathbf{F}$ as

$$\lim_{r\to 0^+} \frac{1}{\pi r^2} \int_{\psi_r} \mathbf{F} \cdot d\mathbf{s}$$

show that the scalar curl of $\mathbf{F}$ at each point $\mathbf{a} \in \mathbb{R}^2 - \{\mathbf{0}\}$ is zero.

(5) Using the definition of curl given in the book (i.e. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$) prove that $\operatorname{curl}\mathbf{F}(\mathbf{a}) = \mathbf{0}$ for all $\mathbf{a} \in \mathbb{R}^2 - \{\mathbf{0}\}$.

Observe that problem E gives an example of a vector field having curl everywhere zero but which is not a conservative vector field and which does not have path independent line integrals. This example will reoccur frequently.