MA 121: Practice Exam 2 **Name:**______________________________

Here are some practice problems. Some of them are more difficult than the problems that will appear on the exam. There are certainly more problems here than will appear on the exam.

(1) Write down the right Riemann sum with $n=5$ rectangles for the function $f(x)=\frac{1}{\sqrt{x}}$ on the interval $[1,5]$.

Solution: Recall that a Riemann sum with n rectangles is of the form $\sum_{i=1}^{n} f(c_i)\Delta x$. In our case, we have $\Delta x = 4/n$ and $c_i = 1 + i\Delta x = 1 + 4i/n$. Thus, the right Riemann sum is:

$$\frac{1}{\sqrt{(1+4/5)}}(4/5)+\frac{1}{\sqrt{(1+8/5)}}(4/5)+\frac{1}{\sqrt{(1+12/5)}}(4/5)$$
$$+\frac{1}{\sqrt{(1+16/5)}}(4/5)+\frac{1}{\sqrt{(1+20/5)}}(4/5)$$

(2) Use a left Riemann sum with $n=3$ rectangles to approximate the area between the graph of $f(x)=\sqrt[3]{x}$ on the interval $[0,1]$.

Solution: We have $\Delta x = 1/3$ and $c_i = (i-1)\Delta x = (i-1)/3$. Then the left Riemann sum asked for is:

$$0+\sqrt[3]{(1/3)}(\frac{1}{3})+\sqrt[3]{(2/3)}(\frac{1}{3})$$

(3) Use the definition of the definite integral as the limit of Riemann sums to calculate $\int_0^2 x^2\,dx$. I suggest you use right Riemann sums. You will need to know that $\sum\limits_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

Solution: We have $\Delta x = 2/n$ and $c_i = 2i/n$ and $f(c_i) = 4i^2/n^2$. Then:

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x &= \\
\sum_{i=1}^{n} \frac{4i^2}{n^2}\cdot\frac{2}{n} &= \\
\sum_{i=1}^{n} \frac{8i^2}{n^3} &= \\
\frac{8}{n^3}\sum_{i=1}^{n} i^2 &= \\
\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Taking the limit we get:

$$\begin{aligned}
\int_0^2 x^2\,dx &= \lim_{n\to\infty}\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= \frac{16}{6} \\
&= 8/3.
\end{aligned}$$

(4) Suppose that f is a continuous function. Find $\lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$.

Solution: Since f is continuous, by the EVT, it attains its maximum $M(h)$ on the interval $[x, x+h]$ and also attains its minimum $m(h)$ on $[x, x+h]$. By the properties of integrals, we have:

$$\begin{array}{rcccl} \int_x^{x+h} m(h)\,dt & \leq & \int_x^{x+h} f(t)\,dt & \leq & \int_x^{x+h} M(h)\,dt \\ hm(h) & \leq & \int_x^{x+h} f(t)\,dt & \leq & hM(h). \end{array}$$

Assuming that $h > 0$, dividing both sides by h we have:

$$m(h) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq M(h)$$

So

$$\begin{array}{rcccl} \lim\limits_{h\to 0} m(h) & \leq & \lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt & \leq & \lim\limits_{h\to 0} M(h) \\ f(x) & \leq & \lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt & \leq & f(x). \end{array}$$

because f is continuous and any point in the interval $[x, x+h]$ approaches x as $h \to 0$.

Thus,

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

(5) Suppose that f is a continuous function. State and prove the first version of the fundamental theorem of Calculus.

Statement: FTC: If f is a continuous function on $[a, b]$ then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Proof: We use the definition of the derivative and the previous problem:

$$\begin{array}{rc} \frac{d}{dx}\int_a^x f(t)\,dt & = \\ \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) & = \\ \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt & = \\ f(x) & \end{array}$$

(6) Find the derivative with respect to x of the following functions. For give a rationale for your answer:

(a) $\int_0^x \sin(t^2)\,dt$

Solution: By FTC I, we have

$$\frac{d}{dx}\int_0^x \sin(t^2)\,dt = \sin(x^2).$$

(b) $\int_x^0 \sin(t^2)\,dt$

Solution: Since $\int_x^0 \sin(t^2)\,dt = -\int_0^x \sin(t^2)\,dt$, by the previous problem

$$\frac{d}{dx}\int_x^0 \sin(t^2)\,dt = -\sin(x^2).$$

(c) $\int_{-x}^x \sin(t^2)\,dt$.

Solution: We have:

$$\int_{-x}^x \sin(t^2)\,dt = \int_{-x}^0 \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt.$$

By two problems ago, $\frac{d}{dx}\int_0^x \sin(t^2)\,dt = \sin(x^2)$. We also have $\int_{-x}^0 \sin(t^2)\,dt = \int_0^{-x} f(t)\,dt$.

Let $g(x) = \int_0^{-x} f(t)\,dt$. Then $g(x) = f(u(x))$ where $f(u) = \int_0^u \sin(t^2)\,dt$ and $u(x) = -x$. By FTC I,

$$\frac{df}{du} = \sin(u^2).$$

So by the chain rule:

$$\frac{d}{dx}g(x) = -\sin((-x)^2) = -\sin(x^2)$$

Thus,

$$\frac{d}{dx}\int_{-x}^x \sin(t^2)\,dt = 2\sin(x^2).$$

(7) Prove that if f is a differentiable function such that $f'(x) = 0$ for all x, then f is a constant function.

Solution: We will prove that for all $a < b$, we have $f(a) = f(b)$. Suppose that $a < b$. By the MVT, there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Since $f'(x) = 0$ for all x, we have

$$\frac{f(b)-f(a)}{b-a} = 0.$$

Multiplying both sides by $(b-a)$ produces $f(b) - f(a) = 0$. This means that $f(b) = f(a)$, as desired.

(8) Prove that if $F'(x) = G'(x)$ for all x, then there is a constant C such that $F(x) = G(x) = C$.

Solution: Let $H(x) = F(x) - G(x)$. We have $H'(x) = F'(x) - G'(x) = 0$, for all x. By the previous problem this implies that $H(x)$ is constant. That is, there exists a constant C such that $H(x) = C$. The definition of $H(x)$ then gives us $F(x) = G(x) + C$.

(9) Use only the first version of the fundamental theorem of Calculus and the previous problem to find $\int_0^2 t^2\,dt$.

Solution: By FTC I, the function $F(x) = \int_0^x t^2\,dt$ is an antiderivative of $f(x) = x^2$. By the power rule, we also know that $G(x) = x^3/3$ is an antiderivative of $f(x)$. This means that $F'(x) = G'(x)$. By the previous problem, there exists a constant C such that, for all x,

$$\int_0^x t^2\,dt = \frac{1}{3}x^3 + C.$$

Since, $G(0) = 0$ we have, $0 = \frac{1}{3}(0)^3 + C$, so $C = 0$. Thus,

$$\int_0^x t^2\,dt = \frac{1}{3}x^3$$

for all x. Choosing $x = 2$, we have

$$\int_0^2 t^2\,dt = 8/3.$$

(10) Recalling that $\ln(x) = \int_1^x \frac{1}{t}\,dt$, explain why $\frac{d}{dt}\ln(x) = \frac{1}{x}$. What is the equation of the tangent line to the graph of $y = \ln(x)$ at the point $(e, 1)$?

Solution: By FTC I, $\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$, thus the derivative of $\ln(x)$ is $1/x$. At the point $(e, 1)$, the slope of the tangent line to the graph of $y = \ln(x)$ is $1/e$. The equation of the tangent line is, therefore,

$$y = \frac{1}{e}(x - e) + 1$$

This can be rewritten as

$$y = \frac{1}{e}x.$$

(11) Recall that e^x is defined to be the inverse function of $\ln(x)$. Use this fact and the previous problem to prove that the derivative of $g(x) = e^x$ at $x = 1$ is $g'(1) = e$.

Solution: Since e^x is the inverse of $\ln(x)$, its graph can be obtained by reflecting the graph of $y = \ln(x)$ over the line $y = x$. All tangent lines to the graph of $y = \ln(x)$, when reflected, become tangent lines to the graph of $y = e^x$. If $y = mx + b$ is a line, the line obtained by reflecting it over $y = x$ has equation $y = \frac{1}{m}x - \frac{b}{m}$. The line tangent to the graph of $y = \ln(x)$ at the point $e, 1)$ has slope $\frac{1}{e}$ by the previous problem. The reflected line is tangent to the graph of $y = e^x$ at the point $(1, e)$. Our calculation shows that the line has slope $\frac{1}{1/e} = e$. Thus, the derivative of $g(x) = e^x$ at $x = 1$ is e.

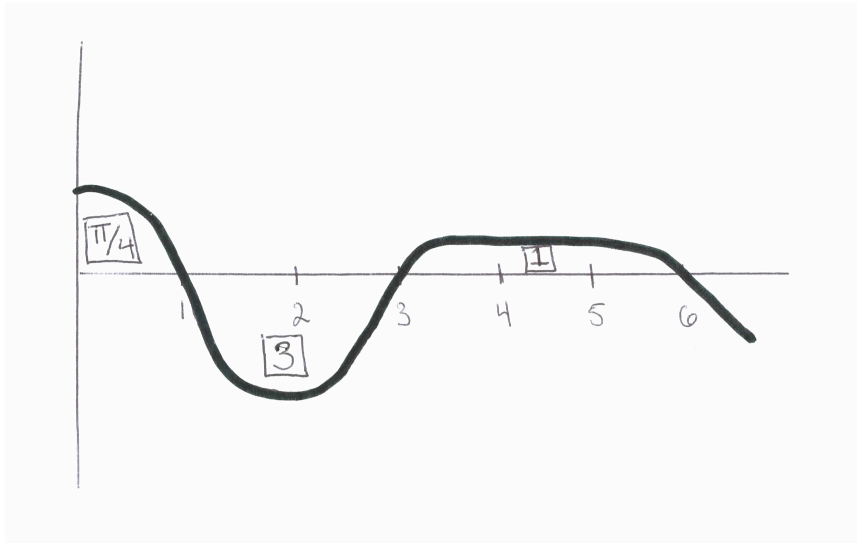
(12) A differentiable one-to-one function $f(x)$ has the properties that $f(1) = 7$ and $f'(1) = 3$. Carefully explain, using pictures, why the derivative of $f^{-1}(x)$ at $x = 7$ is $1/3$.

Solution: You'll need to provide you own pictures. The essence is that the graph of $f(x)$ has a tangent line with slope 3 at the point $(1, 7)$. Reflecting everything over the line $y = x$, converts the line into a tangent line for the graph of $f^{-1}(x)$ at the point $(7, 1)$ with slope $1/3$.

(13) Prove that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.

Solution: The previous two problems demonstrate that if a function $f(x)$ has a tangent line with slope m at the point $(a, f(a))$, then the inverse function $f^{-1}(x)$ has a tangent line with slope $1/m$ at the point $(f(a), a)$. Let $f(x) = \sin x$. At the point $(a, \sin(a))$, the function $f(x)$ has tangent line with slope $\cos(a)$. Thus, at the point $(\sin(a), a)$, the function $\arcsin(x)$ has a tangent line with slope $1/\cos(a)$. Let $x = \sin(a)$. The sides of a right triangle with angle a are labelled x (opposite from a) and $\cos(a)$ (adjacent to a) when the hypotenuse has length 1. By the Pythagorean theorem, we have $\cos(a) = \sqrt{1-x^2}$. Thus, the derivative of $\arcsin(x)$ is $\frac{1}{\cos a} = \frac{1}{\sqrt{1-x^2}}$.

(14) Drawn below is the graph of a function $f(x)$, with the area between the graph and the x axis marked for various intervals. Suppose that $F(x)$ is an anti-derivative of $f(x)$ with $F(0) = 6$. Find $F(6)$.



Solution: Recall that by FTC I, the function $G(x) = \int_0^t f(t)\,dt$ is an antiderivative of $f(x)$. Thus all antiderivatives are of the form $\int_0^t f(t)\,dt + C$ for a constant C. Since $G(0) = 0$, to have an antiderivative F with $F(0) = 6$, we need to let $C = 6$. Let $F(x) = G(x) + 6$. Since $G(x)$ measure the signed area between the graph of $f(t)$ and the t axis for $0 \le t \le x$, we simply plug $x = 6$ in to find $G(6) = (\pi/4) - 3 + 1 = -2 + \pi/4$. Hence, $F(6) = 4 + \pi/4$.

(15) Let $F(x) = \int_0^x t(t+2)(t-1)\,dt$. Find and classify all critical points of $F(x)$.

Solution: By FTC I, $F'(x) = x(x+2)(x-1)$. We have $F'(x) = 0$ at $x = 0, -2, +1$. If $x < -2$, then $F'(x) < 0$. If $-2 < x < 0$, then $F'(x) > 0$. If $0 < x < 1$, then $F'(x) < 0$. If $x > 1$, then $F'(x) > 0$. Thus, F is decreasing on $(-\infty, -2)$ and $(0, 1)$ and increasing on the intervals $(-2, 0)$ and $(1, \infty)$. Thus, $x = -2$ and $x = 1$ are local minima and $x = 0$ is a local maximum.

(16) Using the fact that $\frac{d}{dt}\sin t = \cos t$, prove that $\frac{d}{dt}\cos t = -\sin t$.

Solution: By examining the graphs, we recall that $\cos t = \sin(t + \pi/2)$. Thus, the derivative of $\cos t$ is just the derivative of $\sin t$ shifted left by $\pi/2$. That is: $\frac{d}{dt}\cos t = \cos(t + \pi/2)$. Shifting the graph of cosine left by $\pi/2$ produces the graph of $-\sin t$, so $\frac{d}{dt}\cos t = -\sin(t)$.

(17) Find derivatives (with respect to t) of the following functions:

(a) $f(t) = \sin t + \cos t$

Solution: $f'(t) = \cos t - \sin t$.

(b) $g(t) = e^t \sin t$

Solution: $g'(t) = e^t \sin t + e^t \cos t$.

(c) $h(t) = \sqrt{e^t - 1}$

Solution: $h'(t) = \frac{e^t}{2\sqrt{e^t - t}}$.

(d) $k(t) = \sqrt{e^{t^2} - 1}$

Solution: $k'(t) = \frac{1}{2\sqrt{e^{t^2}-1}}(e^{t^2})(2t)$.

(e) $m(t) = \frac{\sqrt{t}}{\sqrt{t^2-1}}$

Solution: Rewrite $m(t)$ as

$$m'(t) = t^{1/2}(t^2-1)^{1/2}$$

Let $f(t) = t^{1/2}$ and $g(t) = (t^2-1)^{1/2}$. Calculate $f'(t) = (1/2)t^{-1/2}$ and $g'(t) = (1/2)(t^2-1)^{-1/2}(2t) = t(t^2-1)^{-1/2}$. Now use the product rule:

$$\begin{aligned} m(t) &= f(t)g(t) \\ m'(t) &= f'(t)g(t) + f(t)g'(t) \\ m'(t) &= (1/2)t^{-1/2}(t^2-1)^{1/2} + t^{1/2}(t)(t^2-1)^{-1/2} \end{aligned}$$

(f) $n(t) = \sin(\ln t)$

Solution: $n'(t) = \frac{\cos(\ln(t))}{t}$.

(18) Find $\frac{dy}{dx}$ for the curve $x^3 - y^2 = 1$ at the point $(2, \sqrt{7})$.

Solution: Use implicit differentiatiation to find:

$$3x^2 - 2yy' = 0.$$

Solve for y' to get:

$$y' = \frac{3x^2}{2y}$$

Plug in $x = 2$, $y = \sqrt{7}$ to find that at $(2, \sqrt{7})$:

$$\frac{dy}{dx} = \frac{24}{2\sqrt{7}}.$$

(19) Find the following antiderivatives:

(a) $\int t^2 - \sqrt{t} + \frac{1}{t}\, dt$.

Solution: $\frac{t^3}{3} - (2/3)t^{3/2} + \ln|t| + C$

(b) $\int e^{2t} \sin(e^{2t})\, dt$

Solution: Let $u(t) = e^{2t}$. Then $du = 2e^{2t} dt$ so

$$\begin{aligned} \int e^{2t} \sin(e^{2t})\, dt &= \\ \int (1/2) \sin u\, du &= \\ -(1/2)\cos u + C &= \\ -(1/2)\cos(e^{2t}) + C \end{aligned}$$

(c) $\int \sin^2 t \cos t\, dt$

Solution: Let $u(t) = \sin t$. Then the given antiderivative is:

$$\int u^2\, du = (1/3)u^3 + C = (1/3)\sin^3 t + C.$$

(d) $\int \frac{\arcsin t}{\sqrt{1-t^2}}\, dt$

Solution: Let $u(t) = \arcsin t$. Then $du = \frac{1}{\sqrt{1-t^2}} dt$, so

$$\begin{aligned} \int \frac{\arcsin t}{\sqrt{1-t^2}}\, dt &= \\ \int u\, du &= \\ u + C &= \\ \arcsin t + C \end{aligned}$$

(20) Find the following definite integrals:

(a) $\int_1^3 x^2 - \frac{1}{\sqrt[3]{x}}\, dx$

Solution:

$$(1/3)x^3 - (3/2)x^{2/3}\Big|_1^3 = (9 - (\sqrt[3]{3^5}/2)) - (1/3 - 3/2).$$

(b) $\int_0^{\pi/4} \sin(2t)\, dt$

Solution: $-\cos(t)/2\Big|_0^{\pi/2} = 0 + \frac{1}{2}$.

(c) $\int_0^8 te^{t^2}\, dt$

Solution: Let $u = t^2$. Then $du = 2t\,dt$ and $u(0) = 0$ and $u(8) = 64$ so

$$\int_0^8 te^{t^2}\,dt = \frac{1}{2}\int_0^{64} e^u\,du = \frac{1}{2}(e^{64}-1).$$

(d) $\int_0^3 t\sin(t)\,dt$

Solution: Let $u = t$ and $dv = \sin(t)dt$. Then $du = dt$ and $v = -\cos t$. So using integration by part:

$$\begin{aligned} \int_0^3 t\sin t\,dt &= -t\cos t\Big|_0^3 - \int_0^3(-\cos t)\,dt \\ &= -3\cos(3)+\sin(3). \end{aligned}$$

(21) Prove that every function $f(x)$ having the property that $f'(x)$ is proportional to $f(x)$ is of the form $f(x) = Ae^{kx}$ for some constants A and k. Suppose that a population of bacteria doubles every 1/2 hour. Explain the relevance of the the equation $f'(x) = kf(x)$ to determining the population of the bacteria after 75 minutes.

Solution: Since $f'(x)$ is proportional to $f(x)$ we have, for some constant k:

$$f'(x) = kf(x).$$

If $f(x) \neq 0$, we have:

$$\frac{1}{f(x)}f'(x) = k.$$

Integrate both sides using substitution to get:

$$\begin{aligned} \int \tfrac{1}{f(x)}f'(x)\,dx &= \int k\,dx \\ \ln|f(x)| &= kx + C \end{aligned}$$

Exponentiate both sides to get:

$$\pm f(x) = e^{kx+C}$$

Since $e^{kx+C} = e^{kx}e^C$, we let $A = e^C$ and write:

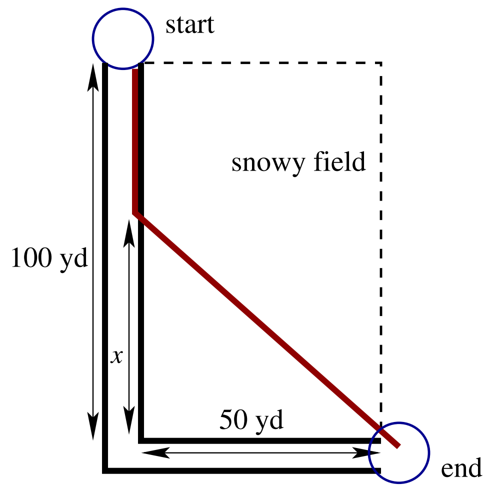
$$f(x) = \pm Ae^{kx}$$

If we allow A to be positive or negative, we can write

$$f(x) = Ae^{kx}$$

Allowing A to also be zero, gives us the solution when $f'(x)$ is zero. We don't address why this is the unique solution with $f'(x) = 0$ for some zero.

(22) In walking from Colby to my house I need to walk by the Mount Merici school. See the diagram below. I can cut straight across the field through deep snow, to my house, or I can walk along the road with no snow. Of course, I can also walk along the road for some distance, and then cut through the field.



If I walk at 90 yards per minute with no snow and at 30 yards per minute through deep snow, what path gets me to my house in the shortest amount of time? That is, how far should I walk along the road, before cutting through the snow? Solve this problem in several steps:

(a) Suppose that I choose to start cutting through the field, x yards before the corner. How much time do I spend walking on the road?

Solution: The distance walked in the road is $(100-x)$ yards. I walk at 90 yd per minute and so the time spent on the road is $(100-x)/90$.

(b) Suppose that I choose to start cutting through the field, x yards before the corner. What distance do I travel through the field? How much time do I spend walking through the field?

Solution: By the Pythagorean theorem, the distance walked in the snowy field is $\sqrt{50^2+x^2}$ yd. I walk at a rate of 30 yd per min in the snow so the time spent in the field is $\frac{1}{30}\sqrt{50^2+x^2}$.

(c) Let $s(x)$ be the total time spent getting to my house from the start of Mt. Merici's driveway, where I choose to start cutting through the field x yards before the corner. Find a formula for $s(x)$.

Solution: $s(x) = \frac{100-x}{90} + \frac{1}{30}\sqrt{50^2+x^2}$.

(d) Find the critical points of $s(x)$. Is one of these the global minimum? If not what is?

Solution: Notice that $0 \le x \le 100$. We have:

$$s'(x) = -\frac{1}{90} + \frac{x}{30\sqrt{50^2+x^2}}.$$

We find the critical points:

$$\begin{array}{rcl} -\frac{1}{90} + \frac{x}{30\sqrt{50^2+x^2}} & = & 0 \\ -1 + \frac{3x}{\sqrt{50^2+x^2}} & = & 0 \\ \sqrt{50^2+x^2} & = & 3x \\ 50^2+x^2 & = & 9x^2 \\ \frac{50^2}{8} & = & x^2 \\ \frac{50}{2\sqrt{2}} & = & x \\ 25/\sqrt{2} & = & x. \end{array}$$

(We only need a positive value for x.)

Notice that $s'(x) < 0$ only if $x < 25/\sqrt{2}$ and $s'(x) > 0$ if $x > 25/\sqrt{2}$. Thus $x = 25/\sqrt{2}$ is a local minimum. Plug into $s(x)$ to find

$$s(25/\sqrt{2}) = \frac{100-25\sqrt{2}}{90} + \frac{1}{30}\sqrt{50^2+50^2/8}$$

This means that

$$s(25/\sqrt{2}) \approx .718 + 1.768 \approx 2.486$$

We also check the endpoints $s(0) \approx 2.778$ and $s(100) \approx 3.727$ minutes. We conclude that starting to cut across the field at $x = 25/\sqrt{2}$ yd from the corner will minimize the time spent walking home.