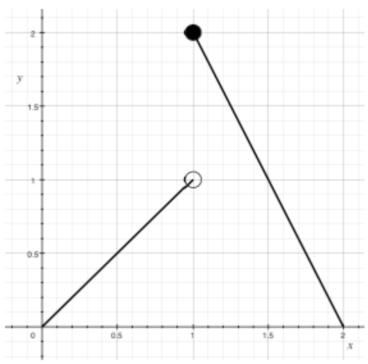
MA 121: Practice Exam 1 Solutions **Name:**____________________

Here are some practice problems. Some of them are more difficult than the problems that will appear on the exam. There are certainly more problems here than will appear on the exam.

(1) Here is the graph of a function $f(x)$. Use both words and pictures to give an $\varepsilon - \delta$ argument that $\lim_{x \to 1} f(x) \neq 1.5$.



Solution: Choose $\varepsilon = .25$. No matter what vertical lines are drawn on the sides of $x = 1$, it is impossible to make the graph between those horizontal lines also be between the lines $y = 1.5 + \varepsilon = 1.75$ and $y = 1.5 - \varepsilon = 1.25$. Thus, for no $\delta > 0$ is it the case that if $1 - \delta < x < 1 + \delta$ then $1.5 - \varepsilon < f(x) < 1.5 + \varepsilon$. This means that $\lim_{x \to 1} f(x)$ does not exist.

(2) Use the limit definition of the derivative to prove that the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Solution: By definition:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \tfrac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \tfrac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \tfrac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \tfrac{2hx + h^2}{h} \\ &= \lim_{h \to 0} 2x + h \\ &= 2x. \end{aligned}$$

The final equality holds since as h gets close to 0, $2x+h$ gets close to $2x$.

(3) Use the limit definition of the derivative to prove that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

Solution: By definition:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \tfrac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \tfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \tfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \tfrac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \tfrac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \tfrac{1}{\sqrt{x}+\sqrt{x}} \\
&= \tfrac{1}{2\sqrt{x}}.
\end{aligned}$$

(4) Show that $f(x) = x|x|$ is differentiable at $x = 0$. (Hint: begin by rewriting f as a piecewise function.)

Solution: The function $f(x)$ can be rewritten as:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}.$$

If $h < 0$, then

$$\frac{f(0+h)-f(0)}{h} = \frac{-h^2}{h} = -h.$$

Hence,

$$\lim h \to 0^- \frac{f(0+h)-f(0)}{h} = 0.$$

If $h > 0$, then

$$\frac{f(0+h)-f(0)}{h} = \frac{h^2}{h} = h.$$

Hence,

$$\lim h \to 0^+ \frac{f(0+h)-f(0)}{h} = 0.$$

Since the right-hand and left-hand limits are equal,

$$f'(x) = \lim h \to 0 \frac{f(x+h)-f(x)}{h}$$

exists and equals 0.

(5) Give a thorough explanation for why $f(x) = |x|$ is not differentiable at $x = 0$.

Solution: Recall that

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

If $h < 0$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1.$$

Hence,

$$\lim h \to 0^{-} \frac{f(0+h) - f(0)}{h} = -1.$$

If $h > 0$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1.$$

Hence,

$$\lim h \to 0^{+} \frac{f(0+h) - f(0)}{h} = 1.$$

Since the right-hand and left-hand limits are not equal,

$$f'(x) = \lim h \to 0 \frac{f(x+h) - f(x)}{h}$$

does not exist.

(6) Find the equation of the linear approximation to $f(x) = \sqrt{x}$ at $x = 4$. Use this to estimate $\sqrt{5}$.

Solution: The equation of the linear approximation to f at $x = 4$ is given by:

$$L(x) = f'(4)(x-4) + f(4).$$

We have $f'(x) = \frac{1}{2\sqrt{x}}$ and so $f'(4) = 1/4$. We also have $f(4) = 2$ and so

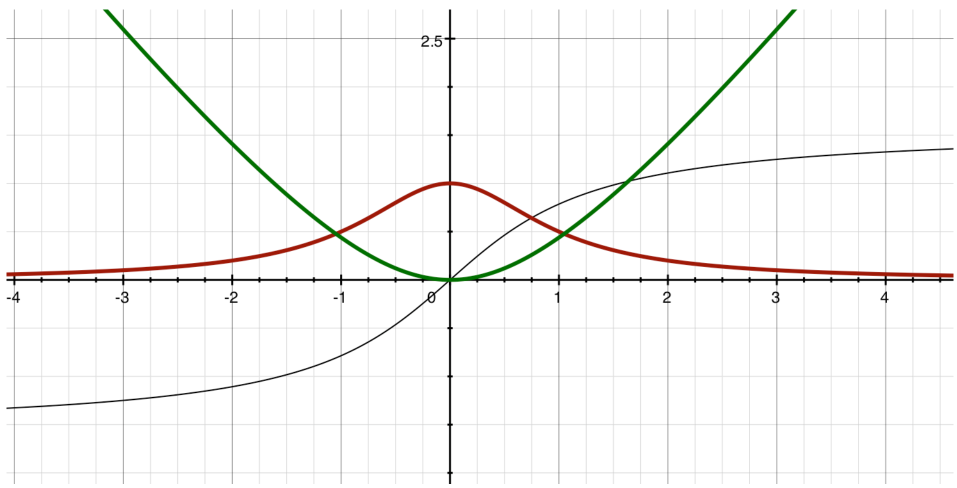
$$L(x) = (1/4)(x-4) + 2.$$

If we plug in $x = 5$ we obtain:

$$\sqrt{5} \approx L(5) = \frac{9}{4}.$$

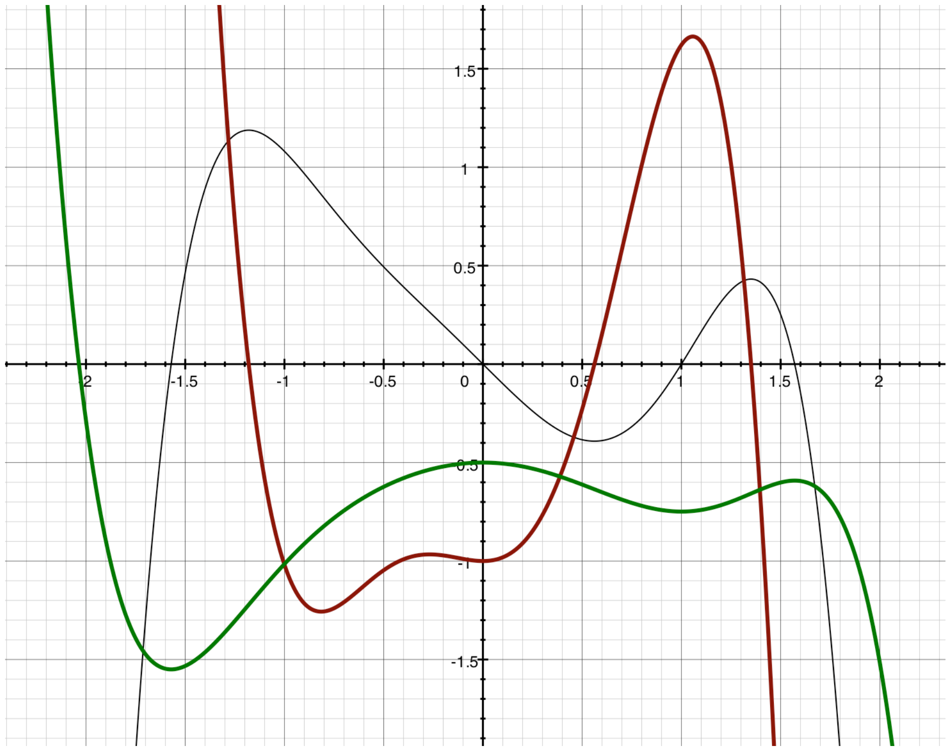
(7) Drawn below is the graph of a function $f(x)$. Sketch both the graph of $f'(x)$ and the graph of a function $g(x)$ such that $g'(x) = f(x)$.

Solution: Superimposed on the original graph are the graph of $f'(x)$ (in red) and the graph of $g(x)$ (in green). The graph of $g(x)$ could be shifted up or down while remaining correct.



(8) Drawn below is the graph of a function $f(x)$. Sketch both the graph of $f'(x)$ and the graph of a function $g(x)$ such that $g'(x) = f(x)$.

Solution: Superimposed on the original graph are the graph of $f'(x)$ (in red) and the graph of $g(x)$ (in green). The graph of $g(x)$ could be shifted up or down while remaining correct.



(9) Hormel is going to manufacture spam cans. Each can will be a cylinder of radius r and height h. The bottom and top of the can are made out of metal that costs 4 cents per square inch. The side of the can is made out of material that costs 2 cents per square inch. The can needs to hold 40 cubic inches of spam. What radius for the can make the cost of materials as low as possible? (Hint: the side of the can has area $2\pi rh$ and the top and bottom each have area πr^2. The volume of the can is $\pi r^2 h$.)

Solution: The cost of the can, which is what we will minimize, is:

$$C = 8\pi r^2 + 4\pi rh$$

The volume of the can is:

$$40 = \pi r^2 h$$

Solving for h we get:

$$h = \frac{40}{\pi r^2}$$

Thus,

$$C = 8\pi r^2 + 4\pi r(\frac{40}{\pi r^2})$$

So,

$$C = 8\pi r^2 + \frac{160}{r}$$

Taking the derivative with respect to C we have:

$$C' = 16\pi r - \frac{160}{r^2}$$

Setting the derivative equal to zero and solving we have:

$$\begin{aligned} 16\pi r - \tfrac{160}{r^2} &= 0 \\ \pi r^3 - 10 &= 0 \\ r^3 &= 10/\pi \\ r &= \sqrt[3]{10/\pi}. \end{aligned}$$

To see that this is a minimum, notice that

$$C'' = 16\pi + \frac{320}{r^3}$$

This is always positive, so C is always concave up which implies that $r = \sqrt[3]{10/\pi}$, being a critical point, is a local minimum.

(10) River Phoenix is going to build a field bordering the Phoenix River. He has 2400 ft of fencing for three sides of the rectangular field and will use the river as the fourth side. What are the dimensions of the field that has the largest area?

Solution: Let h be the length of the sides perpendicular to the river and let l be the length of the side parallel to the river. The amount of fence is

$$2400 = 2h + l.$$

The area of the field is

$$A = hl.$$

We want to maximize A, so plug in for l to get:

$$\begin{aligned} A &= h(2400 - 2h) \\ A &= 2400h - 2h^2. \end{aligned}$$

Calculate the derivative to get:

$$A' = 2400 - 4h.$$

Setting it equal to zero and solving we find $h = 600$. This implies that $l = 1200$. These dimensions give the maximum area since A' is positive (and A is increasing) for $h < 600$ and A' is negative (and A is decreasing) for $h > 600$.

(11) Give complete, precise statements of the Extreme Value Theorem, Mean Value Theorem, and Intermediate Value Theorem.

Solution: See the textbook or your class notes

(12) Draw the graph of a function with domain the interval $[0,1]$ that does not attain a global maximum or minimum on the interval.

Solution: See the textbook or your class notes

(13) Draw the graph of a continuous function with domain the interval $(0,1)$ that does not attain a global maximum or minimum on the interval.

Solution: See the textbook or your class notes

(14) Give the equation of a function that is continuous on the interval $[0,1]$ but does not have a derivative at $x = 1/2$.

Solution: The function $f(x) = |x - 1/2|$ does the trick.

(15) Give the equation of a bounded continuous function on the interval $(0,1]$ that does not have a limit as $x \to 0^+$. (The term "bounded" means that there is a number a such that $|f(x)| \leq a$ for all x in $(0,1]$.)

Solution: The function $f(x) = \sin(1/x)$ for $0 < x \leq 1$ does the trick.

(16) Give the equation of a continuous function on the interval $(0,1]$ that does not attain a global maximum and also does not attain a global minimum.

Solution: The function $f(x) = \frac{\sin(1/x)}{x}$ for $0 < x \leq 1$ does the trick. (Do you see why?)

(17) Use the Intermediate Value Theorem to explain why if f is a continuous function with domain equal to $[a,b]$ and range contained in $[a,b]$ then there exists c in $[a,b]$ such that $f(c) = c$.

Solution: Let $g(x) = f(x) - x$. Since f and $y = x$ are continuous, $g(x)$ is also continuous. The domain of g is $[a,b]$. Since the range of f is contained in $[a,b]$, we have $f(a) \geq a$ which implies that $g(a) \geq 0$. Similarly, $g(b) \leq 0$. Thus, by the intermediate value theorem, there exists c in $[a,b]$ so that $g(c) = 0$. But this is exactly the same as saying that $f(c) = c$.

(18) Give the main idea of how the Extreme Value Theorem can be used to prove the Mean Value Theorem.

Solution: Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). We want to show that there exists a c in (a,b) so that $f'(c)$ is equal to the average value of f on $[a,b]$. Let $L(x)$ be the function whose graph is the secant line to f through the points $(a, f(a))$ and $(b, f(b))$. Define $D(x) = f(x) - L(x)$ and apply the Extreme Value Theorem to D.

(19) A car is travelling on a straight highway so that at time t (minutes) is is a distance of $t^2 - t + 1$ meters from home.

(a) What is the average velocity of the car between $t = 2$ minutes and $t = 5$ minutes?
Solution: Let $f(t) = t^2 - t + 1$. The average value of f on $[2,5]$ is

$$\frac{f(5) - f(2)}{3} = \frac{21 - 3}{3} = 6.$$

(b) Find a t value so that the instantaneous velocity of the car at time t is exactly equal to its average velocity between 2 and 5 minutes.

Solution: The instantaneous velocity of the car at t is

$$f'(t) = 2t - 1.$$

We want to find t so that $f'(t) = 6$. That is we solve

$$2t - 1 = 6$$

to find out that $t = 7/2$.

(20) Find derivatives of the following functions:

(a) $f(x) = x^{\pi - e}$ (here π and e are the usual numbers)
Solution: $(\pi - e)x^{\pi - e - 1}$

(b) $g(x) = \frac{6}{\sqrt{x}} - \sqrt[3]{x^2}$
Solution: $g(x) = 6x^{-1/2} - x^{2/3}$ so $g'(x) = -3x^{-3/2} - (2/3)x^{-1/3}$.

(c) $k(t) = 15t^3 + 2t^2 - \frac{1}{t}$
Solution: $k'(t) = 45t^2 + 4t + \frac{1}{t^2}$.

(21) For the following functions find, **using calculus**:

(i) All local maxima and minima in the specified interval
(ii) All global maxima and minima in the specified interval
(iii) All x values where the function is decreasing and concave down.
(iv) All x values where the function is decreasing and concave up.

(a) $g(t) = |3t - 4|$ for $-2 \leq t \leq 2$.

Solution:

$$g'(t) = \begin{cases} 3 & x > 4/3 \\ -3 & x < 4/3 \end{cases}.$$

and $g''(t) = 0$ if $t \neq 4/3$.

i) g has a local minimum at $t = 4/3$.

ii) g has a global minimum at $t = 4/3$ and a global maximum at $t = -2$.

iii) Since $g'(t)$ is constant, it is decreasing everywhere $t \neq 4/3$. Thus g is concave down except at $t = 4/3$. The function g is decreasing on the interval $[-2, 4/3)$.

iv) Similar to iii), the function is decreasing and concave up on $[4/3, 2]$.

(b) $f(t) = t + 1/t$.

Solution: We have:

$$\begin{aligned} f'(t) &= 1 - 1/t^2 \\ f''(t) &= 2/t^3. \end{aligned}$$

Consequently, f has critical points at $t = \pm 1$. Since $f''(-1) = -2$, $t = -1$ is a local maximum. since $f''(1) = 2$, $t = 1$ is a local minimum. As $t \to \pm\infty$, the function $f(t) \to \pm\infty$ so f has no global maximum or minimum. The function f is concave down if $t < 0$ and concave up if $t > 0$. It is decreasing if $f'(t) < 0$ and increasing if $f'(t) > 0$. Thus, f is decreasing if $t^2 < 1$. This happens if and only if $-1 < t < 1$. Thus, f is concave down and decreasing if and only if $-1 < t < 0$ and it is concave down and decreasing if and only if $0 < t < 1$.