MA 121: HW 11 Solutions

Answer these questions on a separate sheet of paper.

(1) In class we came up with a formula for the average value of a function $f(x)$ on an interval $[a,b]$. Explain how we did this.

Solution: Let $V(f)$ denote the average value of f on $[a,b]$. We can approximate $V(f)$ by choosing n equally spaced points $c_1, c_2, \ldots, c_n$ such that the distance between them is Δx. Then $V(f)$ is approximately

$$V(f) \approx \frac{1}{n}\sum_{i=1}^{n} f(c_i)$$

If we take the limit as $n \to \infty$, we get the average value:

$$V(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(c_i).$$

We have $\Delta x = (b-a)/n$ so $\frac{1}{n} = \frac{\Delta x}{b-a}$. Substituting in we have

$$V(f) = \lim_{n\to\infty} \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(c_i)$$

We rearrange this to:

$$V(f) = \frac{1}{b-a}\lim_{n\to\infty} f(c_i)\Delta x.$$

The limit of the sums can be written as an integral since integrals are limits of Riemann sums. This gives us:

$$V(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(2) The ideal gas law says that if temperature is held constant, the pressure of a gas is inversely proportional to its volume. Suppose that at a volume of 2 cubic meters the pressure of the gas is 100 kiloPascals. As the volume changes from 2 cubic meters to .5 cubic meters, what is the average pressure of the gas? what is the average rate of change of the pressure of the gas? What is the instantaneous rate of change of the pressure of the gas (as a function of the volume)?

Solution: The ideal gas law gives us $P = \frac{k}{V}$ where k is some constant. When $V = 2$ we have $P = 100$ so $k = 200$. Using the formula for average value we get that:

$$V(P) = \frac{1}{.5-2}\int_2^{.5} \frac{200}{V}\,dV$$

Thus, the average value of pressure is

$$V(P) = -\frac{2}{3}\Big(200\ln(.5) - 200\ln(2)\Big).$$

The average rate of change of the pressure is simply

$$\frac{P(.5)-P(2)}{.5-2} = \frac{(200/.5)-(200/2)}{-(3/2)}.$$

The instantaneous rate of change of pressure is

$$\frac{d}{dV}P(V) = -200/V^2.$$

(3) Suppose that $f(x)$ is a positive function. Let S be the area between the graph of $y = f(x)$ for $0 \le a \le x \le b$ and the x-axis. In class we came up with a formula (involving an integral) for the volume of the solid obtained by rotating S around the y axis. Using the same method find a formula for the volume of the object obtained by rotating S around the x-axis. (You could of course look up the formula. I'm asking you to come up with it on your own using a method similar to what we did in class.)

Solution: Approximate the area under f with n small rectangles, each of width Δx. Assume that the height of the ith rectangle is $f(c_i)$. These rectangles together approximate the area underneath f, so revolving them around the x-axis creates objects that approximate the volume of S. Revolving the ith rectangle around the x-axis creates a solid cylinder. The base of the cylinder has area $\pi f(c_i)^2$ and height Δx. Thus, the volume of S is approximately $\sum_{i=1}^{n}\pi f(c_i)^2\Delta x$. The volume of S is exactly $\lim_{n\to\infty}\sum_{i=1}^{n}\pi f(c_i)^2\Delta x$. We recognize this as a Riemann sum and so write the volume of S as $\int_a^b \pi f(x)^2\,dx$.

(4) The force exerted by a particle with charge +1 at the origin on another particle P of charge +1 a distance of r away is proportional to $\frac{1}{r^2}$.

(a) Calculate the work necessary to move P (in a straight line) from a distance of $r = 2$ to a distance of $r = 1$.

Solution: The work is simply the integral of force. We know that the force required to move a particle at r is $F(r) = \frac{k}{r^2}$ for some constant of proportionality k. Thus, the work required is

$$\int_2^1 \frac{k}{r^2}\,dr = \frac{-k}{r}\Big|_2^1 = -k + \frac{k}{2} = -\frac{k}{2}.$$

(b) Calculate the work necessary to move P from a distance of $r = 2$ to a distance of $r = s$ for some $0 < s$. Call this number $W(s)$.

Solution: It is simply

$$\int_2^s \frac{k}{r^2}\,dr = \frac{-k}{s} + \frac{k}{2}.$$

(c) Find $\lim_{s\to 0^+} W(s)$. Explain what the result means in the language of physics.

Solution: We have

$$\lim_{s\to 0^+} W(s) = \lim_{s\to 0^+} \frac{-k}{s} + \frac{k}{2} = \pm\infty$$

Thus we could never bring the two particles together since it would require an infinite amount of work.

(d) Find $\lim_{s\to\infty} W(s)$. Explain what the result means in the language of physics.

Solution: We have

$$\lim_{s\to infty} W(s) = \lim_{s\to\infty} \frac{-k}{s} + \frac{k}{2} = \frac{k}{2}$$

Thus for any really large value of s the work required to move the particle from 2 to s is approximately $\frac{k}{2}$.

(5) A 100 foot long rope that weighs (1/3) lb per foot is hanging off a 60 foot tall building. One end of the rope is attached to the top of the building and the rope hangs straight down the edge of the building so that 40 feet of rope is coiled at the base of the building. Calculate the work necessary to haul the rope to the top of the building (coiling it at the top as you haul it up). (Hint: Treat the 40 feet of rope coiled on the ground separately from the rope hanging down the side of the building.)

Solution: We compute separately the work required to move the stretched rope to the top of the building and the work required to move the coiled rope to the top of the building. First the stretched rope:

Consider a small piece of rope of length Δx at height c_i above the ground. The force required to move that piece of rope is $(1/3)\Delta x$. The work required to move it to the top of the building is $(1/3)\Delta x(60 - c_i)$. Thus the work required to move the stretched rope to the top of the building is approximately $\sum_{i=1}^{n}(1/3)(60-c_i)\Delta x$ where n is the total number of small pieces of rope. The work is exactly

$$\lim_{n\to\infty}\sum_{i=1}^{n}(1/3)(60-c_i)\Delta x$$

which we recognize as the integral

$$\int_0^{60}(1/3)(60-x)\,dx = \int_0^{60} 20\,dx - \int_0^{60} x/3\,dx.$$

This equals

$$120 + \frac{x^2}{6}\bigg|_0^{60} = 720.$$

All parts of the coiled rope require the same force and move the same distance, so the work required for moving the coiled rope to the top of the building is exactly (1/3)(40)(60) = 800. Thus the total work required to move the entire rope to the top of the building is 800 + 720 = 1520 pound feet.

(6) Consider a cubical tank with sides of length 20 m^3. It is half filled with oil (which has a density of 800 kg/m^3). How much work is required to pump the oil out the top of the tank?

To answer this question, work through the following:

(a) Choose coordinates so that the bottom of the tank is at $y = 0$ and the top is at $y = 20$. We will begin by calculating the work necessary to pump a thin slab of oil out of the tank.

(b) Consider a slab (horizontal cross-section) of oil of thickness Δy and with the bottom of the slab at height $y = c_i$. What is the volume of this slab?

Solution: $400\Delta y$.

(c) What is the mass of the slab of oil in the previous part? Using Newton's second law (F = ma) calculate the force necessary to move the slab upward. (Recall that the force due to gravity is approximately $g = -9.8m/s^2$.) How much work is required to move this slab to the top of the tank.

Solution: The mass of the slab of oil is $(800)(400)\Delta y$. The force required to move it is $(9.8)(800)(400)\Delta y$. It moves a distance of $20 - c_i$. Thus the work required to move the slab is $(9.8)(800)(400)(20 - c_i)\Delta y$.

(d) Write down a sum giving an approximation to the total amount of work necessary to pump the oil out of the tank.

Solution: $\sum_{i=1}^{n}(9.8)(800)(400)(20 - c_i)\Delta y$.

(e) Realize that the total work required is a limit of the previous sum.

Solution: $\lim_{n\to\infty}\sum_{i=1}^{n}(9.8)(800)(400)(20 - c_i)\Delta y$.

(f) Recognize the limit of Riemann sums as being an integral. Write down the integral and solve it using the fundamental theorem of calculus.

Solution: The work is equal to $\int_0^{10}(9.8)(800)(400)(20-y)\,dy$.

(7) Consider the conical tank pictured below that is filled to a height of 10 meters with oil. Calculate the work necessary to pump the oil out the top of the tank.

Solution: The steps are the same as in the previous problem, but we need to calculate the volume of the slab differently.

The slab at height c_i is approximately a cylinder of height Δy and radius $\frac{12.5}{20}c_i$ (you can deduce this from an equation for line formed by the edge of the picture of the cone.) Its volume is therefore approximately $\left(\frac{12.5}{20}c_i\right)^2 \Delta y \approx .39c_i^2\Delta y$ The mass is therefore $(800)(.39)c_i^2\Delta y$. The force required to move the slab is therefore $(9.8)(800)(.39)c_i^2\Delta y$. It moves a distance of $(20-c_i)$ and so the work required to move the slab is $(9.8)(800)(.39)c_i^2(20-c_i)\Delta y$. Adding these up and taking a limit as before produces the limit of Riemann sums that is equal to the work required to pump the oil out. This limit of Riemann sums equals the integral

$$\int_0^{10}(9.8)(800)(.39)y^2(20-y)\,dy = 3057.6\left(\int_0^{10} 20y^2\,dy - \int_0^{10} y^3\,dy\right)$$

This equals $3057.6\left(\frac{20}{3}(10)^3 - \frac{1}{4}(10)^4\right)$.