MA 121: Final Exam Review **Name:**________________________________

The course has emphasized the interplay between the theory of calculus and the ability to do calculations. Once we understand this interplay we can begin to apply calculus to the study of real world phenomena. This process of applying calculus is necessarily incomplete because the real world is complex and even those phenomena which can be understood quantitatively usually require multiple independent variables in their description.

The study guide below is divided into three sections corresponding to the theory of calculus, the calculations in calculus, and the applications of calculus. As you review each of these topics, remember to think about their interaction. If you find particular sorts of problems challenging, try to isolate the problem: is it that you don't understand the theory, that you can't do the calculation, or that you don't see how to translate the problem into mathematics. At the end of each review section, there is a "trouble-shooting guide". Be sure to read and internalize it.

Below, if a result is marked with (*) you should be able to prove the result.

1. THE THEORY OF CALCULUS

1.1. **Limits.**

- The concept of limit expresses the idea that as the inputs to a function approach a certain value, the outputs may be approaching a certain value.
- The formal definition of "limit" is:

  $\lim_{x\to a} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$
  such that if $x \neq a$ and $|x-a| < \delta$ then $|f(x)-L| < \varepsilon$.

  In words we say that "the limit " of a function f as x approaches a is equal to L if by making x close to (but not equal to) a, the numbers $f(x)$ will be as close as we wish to L.
- A continuous function is a function such that for all a, $\lim_{x\to a} f(x) = f(a)$.

- (Extreme Value Theorem) If a continuous function is defined on a closed and bounded interval, then the function attains a maximum value and a minimum value on that interval.
- (Intermediate Value Theorem) If a continuous function is defined on a closed and bounded interval, then the function attains all values between the values taken at the endpoints of the interval.

1.2. **Derivatives.**

- The derivative of $f(x)$ is defined as
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
if the limit exists
- The derivative $f'(x)$ can be interpreted as the slope of the tangent line to the graph of f at $(x, f(x))$ or as the instantaneous rate of change of f at x.
- (*) (Mean Value Theorem) If a function f is continous on a closed bounded interval $[a,b]$ and differentiable on the open interval (a,b) then there exists a value c between a and b so that
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$
- If $f'(x) > 0$ then f is increasing. If $f'(x) < 0$ then f is decreasing.
- If x is a local maximum or minimum of f, then either $f'(x) = 0$ or $f'(x)$ does not exist.
- If $f''(x) < 0$ then f is concave down. If $f''(x) > 0$, then f is concave up.
- If $f'(x) = 0$ and if $f''(x) < 0$, then x is a local maximum. If $f'(x) = 0$ and if $f''(x) > 0$, then x is a local minimum.
- (*) If $f'(x) = 0$ for all x values, then f is constant.
- (*) If $f'(x) = g'(x)$ for all x values, then there is a constant C such that $f(x) = g(x) + c$.

1.3. **Integrals.**

- If f is defined on the interval $[a,b]$, then an nth Riemann sum of f is defined by:

(1) subdivide the interval $[a,b]$ into n equal subintervals, each of length Δx.
(2) choose a point c_i in the ith subinterval
(3) summing for each i such that $1 \leq i \leq n$, the quantities $f(c_i)\Delta x$.

That is, an nth Riemann sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

- The nth right Riemann sum is obtained by choosing each c_i to be the right endpoint of the ith subinterval. That is, $c_i = a + i\Delta x$.
- The nth left Riemann sum is obtained by choosing each c_i to be the left endpoint of the ith subinterval. That is, $c_i = a + (i-1)\Delta x$.
- The definite integral of f on the interval $[a,b]$ is defined to be the limit of Riemann sums. That is,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

- Riemann sums approximate signed area between a graph and the x-axis. Definite integrals give the exact signed area between the graph and the x-axis.

1.4. **The Fundamental Theorem of Calculus.**

- (*) The first version of the fundamental theorem of calculus says that for a continuous function f and a a constant in the domain of f:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

- Informally, the first version of FTC says that the rate of change of the area between the graph of a function and the x axis is equal to the function itself.
- (*) The second version of the fundamental theorem of calculus says that if $F(x)$ is an antiderivative of the continuous function $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- Informally, the second version of FTC says the area between the graph of f and the x-axis can be determined by evaluating an antiderivative of f on the endpoints of the interval.

2. CALCULATIONS

- We know how to calculate limits of various sorts of functions by looking at their equations and at their graphs
- We know how to prove that certain functions do or do not have a limit by using the formal definition of limit
- We know how to calculate the derivatives of functions like x, x^2, $\sqrt{x}$, and $1/x$ using the definition of the derivative
- We know how to calculate integrals of functions like x and x^2 using the definition of the definite integral
- We know how to draw graphs of derivatives and antiderivatives given the graph of a function.
- For the derivative we know the following methods:
  - The power rule
  - The product and quotient rules
  - The chain rule (the most important one!)
  - Derivatives of exponential and logarithm functions
  - Derivatives of trigonometric functions
  - Derivatives of inverse functions
  - Implicit differentiation
- For the (indefinite) integral we know the following methods. Be sure to understand how these methods are related to the methods for derivatives.
  - The power rule
  - Substitution
  - Integration by parts
  - Integrals of exponential functions and $f(x) = \frac{1}{x}$.
  - Integrals of $\sin x$ and $\cos x$
- We know how to solve separable differential equations using substitution
- We know how to draw slope fields

3. APPLICATIONS

- We know how to describe a few real world phenomena using differential equations
- We know how to use Riemann sums to find formulas for average value, volumes, and work (The hard part here is believing we have the right answer!)

- We know how to use implicit differentiation to solve related rates problems. (The hard part here is the set-up – not the calculus!)