MA 302: Practice Exam 3 **Name:**________________________

The final exam is cumulative. The following problems concern only material since Exam 2. You should also use previous exams, practice exams, quizzes, and homework to study.

(1) Find a parameterization of the surface formed by the graph of $z = x^2 - y^2$ with (x, y) in the triangle in the xy-plane formed by the x-axis, the y-axis, and the line $y = -x + 1$.

(2) Is the surface in the previous problem a smooth surface? If no, at what points is it not smooth?

(3) Find a parameterization of the surface formed by rotating the curve $\begin{pmatrix} \cos t + 5 \\ 2\sin t \end{pmatrix}$ with $0 \leq t \leq 2\pi$ around the y-axis.

(4) Consider the surface

$$\mathbf{X}(s,t) = \begin{pmatrix} 2\sin 3t + t \\ \cos 2s \\ t^2 + s^2 \end{pmatrix}, \quad 0 \leq t \leq \pi/4, \quad 0 \leq s \leq \pi$$

Find the tangent and normal vectors to $\mathbf{X}$ at the point $(\pi/6, \pi/6)$. Is the surface smooth?

(5) Suppose that $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, and that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth, oriented surface. Let $h\colon E \to D$ be a smooth, orientation reversing change-of coordinate function. Prove that

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{X}\circ h} \mathbf{F} \cdot d\mathbf{S}.$$

(6) Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a C^1 vector field, and that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth, oriented surface. Let $h\colon E \to D$ be a smooth change-of coordinate function. Prove that

$$\iint_{\mathbf{X}} f\, dS = \iint_{\mathbf{X}\circ h} f\, dS.$$

(7) Suppose that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth, oriented surface with unit normal $\mathbf{n}$. Suppose that $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field. Prove that

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F} \cdot \mathbf{n}\, dS.$$

(8) Use the previous result to integrate the vector field $\mathbf{F}(x,y,z) = (x,y,z)$ over the unit sphere (with outward normal) in $\mathbb{R}^3$.

(9) Let S be the disc of radius 1 centered at $(1,0,0)$ in $\mathbb{R}^3$ which is parallel to the yz-plane. Orient S with normal vector pointing in the direction of the postive x-axis. Use the definition of surface integral to calculate the flux of $\mathbf{F}(x,y,z) = (-xy, yz, xz)$ through S.

(10) Use the same surface S and $\mathbf{F}$ as in the previous problem, but now use Stoke's theorem to calculate the flux of the curl of the previous problem.

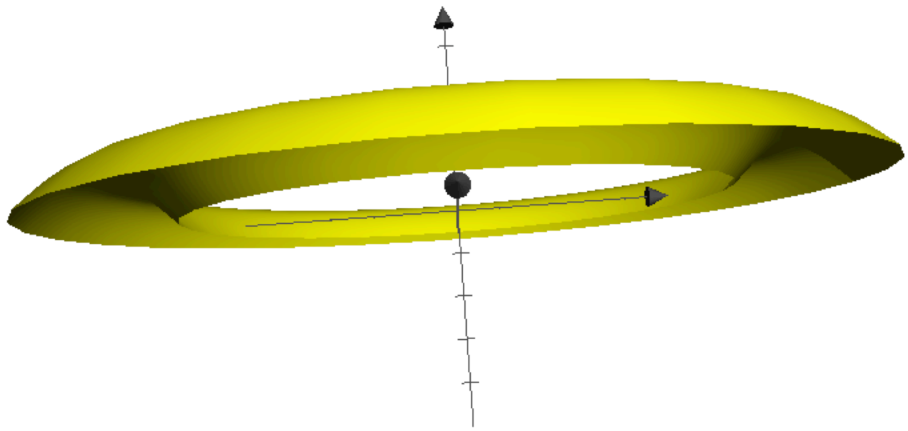
(11) Let $S \subset \mathbb{R}^3$ be an ellipsoid enclosing the origin, oriented outward. Let $P \subset \mathbb{R}^3$ be a cube enclosing the origin and enclosed by S. Orient P outward. Let $\mathbf{F}$ be an incompressible vector field defined on $\mathbb{R}^3 - \{\mathbf{0}\}$. Prove that the flux of $\mathbf{F}$ through P is the same as the flux of $\mathbf{F}$ through S.

(12) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$. Let $\mathbf{a}$ be a point in $\mathbb{R}^3$. For each $n \in \mathbb{N}$, let V_n be a compact 3-dimensional region containing $\mathbf{a}$, such that the regions V_n limit to $\mathbf{a}$. Oriente the boundary of V_n outwards. Use the divergence theorem to prove that

$$\operatorname{div} \mathbf{F}(\mathbf{a}) = \lim_{n\to\infty} \frac{1}{\operatorname{vol} V_n} \iint_{\partial V_n} \mathbf{F} \cdot d\mathbf{S}.$$

(13) Let S be the box with corners $(\pm 1, \pm 1, \pm 1)$, oriented outward. Let $\mathbf{F}(x,y,z) = \begin{pmatrix} xyz \\ xy \\ z \end{pmatrix}$. Find the flux of $\mathbf{F}$ through S.

(14) Let S be a surface formed by rotating the image of $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ \sin t \end{pmatrix}$, $2\pi \le t \le 3\pi$ around the y–axis. Orient S so that at the point $(2\pi + \pi/2, 1, 0)$ there is an upward pointing normal vector. For the following vector fields, find the flux of the vector field through S. (Hint: there are easy ways and there are hard ways...)



(a) $\mathbf{F}(x,y,z) = \begin{pmatrix} x+y \\ -y+z \\ -x+y \end{pmatrix}$

(b) $\mathbf{F}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $\mathbf{F}(x,y,z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(d) $\mathbf{F}(x,y,z) = \frac{1}{x^2+z^2} \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}$

(15) Prove that inside a hollow planet there is no gravity. (You may use Gauss' Law of Gravitation.)

(16) Give a complete, precise statement of both Stokes' theorem and the divergence theorem.