MA 302: Practice Exam 1 **Name:**________________________

(1) Give an example of a vector field $\mathbf{F}$ having $\operatorname{curl}\mathbf{F} = \mathbf{0}$, but where $\mathbf{F}$ is not a gradient field.

(2) Know the formal definitions of the following terms or the complete precise statements of the following theorems:

(a) Green's theorem

(b) planar divergence theorem

(c) conservative vector field

(d) gradient field

(e) potential function

(f) parameterized surface

(g) orientable surface

(h) one-sided surface

(3) Give an example of a one-sided surface in $\mathbb{R}^3$.

(4) Give an example of an orientable surface in $\mathbb{R}^3$.

(5) Prove the following:

(a) Suppose that $D \subset \mathbb{R}^2$ is a type III region and that $\mathbf{F}\colon D \to \mathbb{R}^2$ satisfies the hypotheses of Green's theorem. Prove that the conclusion of Green's theorem holds.

(b) Suppose that $D \subset \mathbb{R}^2$ is the union of two type III regions along a portion of their boundaries. Suppose also that $\mathbf{F}\colon D \to \mathbb{R}^2$ satisfies the hypotheses of Green's theorem. Prove that the conclusion of Green's theorem holds.

(c) Suppose that $D \subset \mathbb{R}^2$ is simply connected and that $\mathbf{F}\colon D \to \mathbb{R}^2$ has $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Prove that $\mathbf{F}$ has path independent line integrals in D.

(d) Suppose that $\mathbf{F}\colon D \to \mathbb{R}^n$ has path independent line integrals. Describe the creation of a potential function for $\mathbf{F}$ and prove that the gradient of this function is $\mathbf{F}$.

(e) Prove that if $\mathbf{F}\colon D \to \mathbb{R}^n$ is a gradient field and if C is a simple closed curve in D, then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(6) Let $D \subset \mathbb{R}^2$ be the region bounded by the graphs of the equations $y = x^3$ and $y = x$ and with $x \geq 0$. Suppose that $\mathbf{F}(x,y) = (xy + y, y^2 x)$.

(a) Is D a type I, II, or III region or none of the above?

(b) Orient ∂D so that D is always on the left. Calculate $\int\limits_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ directly.

(c) Calculate $\iint\limits_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$ directly.

(d) What is the relevance of Green's theorem to the preceding problems?

(e) Is the vector field $\mathbf{F}$ conservative?

(7) Let $\mathbf{F}(x,y,z) = \frac{-1}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Suppose that a particle is located at the point $(1,1,0)$ and moves via the path $\mathbf{x}(t) = (t, \cos t, t \sin t)$ to the point $(\pi/2, 0, \pi/2)$. How much work is done?

(8) What is the flux of the vector field $\mathbf{F}(x,y) = (-y^2 x, x^2 y)$ across the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

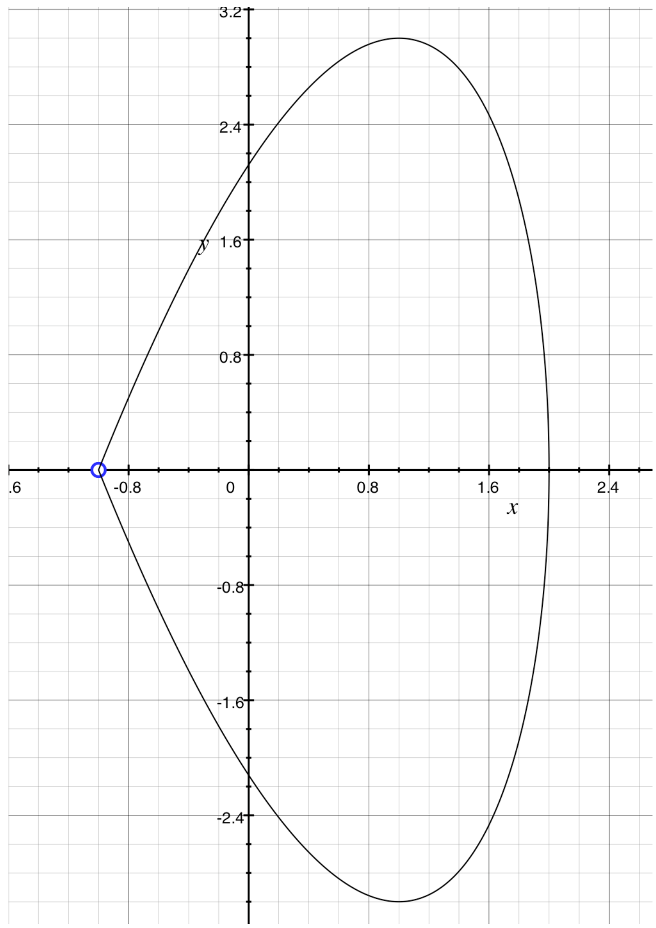
(9) What is the circulation of the vector field $\mathbf{F}(x,y) = (-y^2 x, x^2 y)$ around the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

(10) A wire C is bent into the shape of a circle of radius 1 centered at the origin in $\mathbb{R}^2$. It is given a charge of $+1$ and so generates an electric field $\mathbf{F}$. How much work is done in moving a charged particle from $(1/2, 0)$ to $(0,0)$? Does it matter what path is taken? Why not? (You may leave your answer in integral form.)

(11) Explain why $\operatorname{curl} \mathbf{F}(\mathbf{a}) \cdot \mathbf{k} = \lim_{r \to 0^+} \frac{1}{r^2} \int_{S_r} \mathbf{F} \cdot d\mathbf{s}$ where S_r is a square centered at $\mathbf{a}$ with the distance between midpoints of opposite sides equal to r. $\mathbf{F}$ is a planar vector field.

(12) Explain why $\operatorname{div} \mathbf{F}(\mathbf{a}) = \lim_{r \to 0^+} \frac{1}{r^2} \int_{S_r} \mathbf{F} \cdot \mathbf{n}\, ds$ where S_r is a square centered at $\mathbf{a}$ with the distance between midpoints of opposite sides equal to r. $\mathbf{F}$ is a planar vector field.

(13) Find a single variable integral representing the area enclosed by the path $\phi(t) = (2\cos(2t), 3\sin(3t))$ for $-\pi/3 \leq t \leq \pi/3$.



(14) Let $\sigma\colon [1,2] \to \mathbb{R}^2$ be the path $\sigma(t) = (e^{t-1}, \sin(\pi/t))$. Let $\mathbf{F}(x,y) = (2x\cos y, -x^2\sin y)$. Compute $\int_\sigma \mathbf{F}\cdot d\mathbf{s}$.