MA 302: Practice Exam 1 Solutions **Name:**________________________

(1) Let $F(x, y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F.

Solution:

$$DF(x, y) = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \\ 3 - 2y & -2x \end{pmatrix}$$

(2) Let $F(x, y) = (x - y, x + y)$ and let $G(x, y) = (x \cos y, x \sin y)$. Find the derivative of $F \circ G$ using the chain rule.

Solution:

$$\begin{aligned} DF(x, y) &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ DG(x, y) &= \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix} \\ D(F \circ G)(x, y) &= DF(G(x, y))DG(x, y) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix} \\ &= \begin{pmatrix} \cos y - \sin y & -x \sin y - x \cos y \\ \cos y + \sin y & -x \sin y + x \cos y \end{pmatrix} \end{aligned}$$

(3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0, 1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P?

Solution: The rotation of the P relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) can be described by the path $(\cos(6\pi t + \pi/2), \sin(6\pi t + \pi/2))$. Thus, $\mathbf{x}(t) = (\cos(6\pi t + \pi/2) + t, \sin(6\pi t + \pi/2) + \sin t)$.

(4) A rotating circle of radius 1 follows a helical path in $\mathbb{R}^3$ so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t, the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let

P be the point on the circle which is at $(1,0)$ at time $t=0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P.

Solution: Relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) the point P follows the path $\cos t\mathbf{T} + \sin t\mathbf{N}$ where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and unit normal vectors to $\mathbf{c}(t) = (\sin t, \cos t, t)$ respectively. Those formulae are

$$\begin{aligned} \mathbf{T}(t) &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} \\ \mathbf{N}(t) &= \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{c}(t) &= \cos t\mathbf{T} + \sin t\mathbf{N} + \mathbf{c}(t) \\ &= \tfrac{\cos t}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}. \end{aligned}$$

(5) Explain what it means for curvature to be an intrinsic quantity.

Solution: The curvature of a path $\mathbf{x}(t)$ at t_0, depends only on the curve itself at t_0, not on the parameterization $\mathbf{x}$.

(6) Prove that the curvature at any point of a circle of radius r is $1/r$.

Solution: A circle of radius r can be parameterized as $\mathbf{x}(t) = (r\cos t, r\sin t)$ for $0 \le t \le 2\pi$. We have:

$$\begin{aligned} \mathbf{x}'(t) &= (-r\sin t, r\cos t) \\ ||\mathbf{x}'(t)|| &= r \\ \mathbf{T} &= (-\sin t, \cos t) \\ \mathbf{T}' &= (-\cos t, -\sin t) \\ ||\mathbf{T}'|| &= 1 \\ \kappa(t) &= ||\mathbf{T}'||/||\mathbf{x}'|| \\ &= 1/r. \end{aligned}$$

(7) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \le t \le 5$. Find an integral representing the distance travelled by the particle after t seconds.

Solution: The distance travelled after t seconds is

$$\begin{aligned} s(t) &= \int_1^t ||\mathbf{x}'(\tau)||\,d\tau \\ &= \int_1^t \sqrt{1+4t^2+9t^4}\,d\tau. \end{aligned}$$

(8) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize $\mathbf{x}$ by arc length.

Solution: We compute,

$$s(t) = \int_1^t \sqrt{4\tau^2 + 36\tau^2}\,d\tau = \int_1^t 2\tau\sqrt{10}\,d\tau. = \sqrt{10}(t^2-1).$$

Then,

$$s^{-1}(t) = \sqrt{t/\sqrt{10}+1}$$

Consequently,

$$\mathbf{y}(t) = \mathbf{x}\circ s^{-1}(t) = (t/\sqrt{10}+1, 3t/\sqrt{10}+3)$$

is the reparameterization of $\mathbf{x}$ by arclength.

(9) Suppose that $\mathbf{x}(t)$ is a path in $\mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, $\mathbf{x}$ is a path joining $\mathbf{a}$ to $\mathbf{b}$.) Find a path which has the same image as $\mathbf{x}$ but which joins $\mathbf{b}$ to $\mathbf{a}$.

Solution: $\mathbf{y}\colon [-1,0] \to \mathbb{R}^n$ defined by $\mathbf{y}(t) = \mathbf{x}(-t)$ will do the trick since $\mathbf{y}(0) = \mathbf{a}$ and $\mathbf{y}(-1) = \mathbf{b}$.

(10) Let $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t. Let $\mathbf{y} = \mathbf{x}\circ\phi$ be an orientation reversing reparameterization of $\mathbf{x}$. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f\,ds = \int_{\mathbf{x}} f\,ds$.

Solution: Since ϕ is orientation reversing, $|\phi'(t)| = -\phi'(t)$. Hence, $||\mathbf{y}'(t)|| = -||\mathbf{x}'(\phi(t))||\phi'(t)$. Thus,

$$\int_{\mathbf{y}} f\,ds = -\int_c^d f(\mathbf{x}(\phi(t)))||\mathbf{x}'(\phi(t))||\phi'(t)\,dt.$$

Substitute $u = \phi(t)$ and $du = \phi'(t)dt$ to get:

$$\int_{\mathbf{y}} f\,ds = -\int_b^a f(\mathbf{x}(u))||\mathbf{x}'(u)||\,du.$$

Reversing the limits of integration eliminates the negative sign and so the result follows.

(11) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x,y) = y\cos x$. Let $F(x,y) = (-y, x)$. Find one-variable integrals representing $\int_{\mathbf{x}} f\,ds$ and $\int_{\mathbf{x}} F\cdot d\mathbf{s}$.

Solution: Notice that

$$\begin{aligned}
\mathbf{x}(t) &= (t\cos t, t\sin t)\\
\mathbf{x}'(t) &= (\cos t - t\sin t, t\cos t + \sin t)\\
||\mathbf{x}'(t)|| &= \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}
\end{aligned}$$

Thus,

$$\int_{\mathbf{x}} f\,ds = \int_0^{2\pi} t\sin t\cos(t\cos t)\sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}\,dt.$$

And,

$$\begin{aligned}
\int_{\mathbf{x}} F\cdot ds &= \int_0^{2\pi} \begin{pmatrix} -t\sin t\\ t\cos t\end{pmatrix}\cdot\begin{pmatrix}\cos t - t\sin t\\ \sin t + t\cos t\end{pmatrix}\,dt.\\
&= \int_0^{2\pi} t^2\,dt\\
&= 8\pi^3/3.
\end{aligned}$$

(12) The gravitation vector field in $\mathbb{R}^3$ is $F(\mathbf{x}) = \mathbf{x}/||\mathbf{x}||^3$. Find an integral representing the amount of work required to move an object through the vector field F along the path $\mathbf{x}(t) = (t\cos t, t\sin t, t)$ for $1 \leq t \leq 2\pi$.

Solution: Notice that $\mathbf{x}'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$ and $||\mathbf{x}'(t)|| = t\sqrt{3}$. Thus, the work required is

$$\int_1^{2\pi} F(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt = \int_1^{2\pi}\frac{1}{3^{3/2}t^2}\begin{pmatrix}\cos t\\ \sin t\\ 1\end{pmatrix}\cdot\begin{pmatrix}\cos t - t\sin t\\ \sin t + t\cos t\\ 1\end{pmatrix}\,dt.$$

This last integral is equal to $\int_1^{2\pi}\frac{2}{3^{3/2}t^2}\,dt$.That is equal to $2/3^{3/2} - 1/(3^{3/2}\pi)$.

(13) Let $F(x,y) = (ax+by, cx+dy)$ be a transformation of space, where a,b,c,d are constants such that $ad - bc \neq 0$. Suppose that an object is moving in a circle $\mathbf{x}(t) = (\cos t, \sin t)$. Let $\mathbf{y}(t) = F(\mathbf{x}(t))$. If all forces stop acting on an object following the path $\mathbf{y}$ at time $t = \pi$, where will the object be 3 seconds later?

Solution: We find that $\mathbf{y}(\pi) = F(\mathbf{x}(\pi)) = (-a, -c)$. By the chain rule,

$$\mathbf{y}'(t) = DF\,D\mathbf{x} = (-a\sin t + b\cos t, -c\sin t + d\cos t).$$

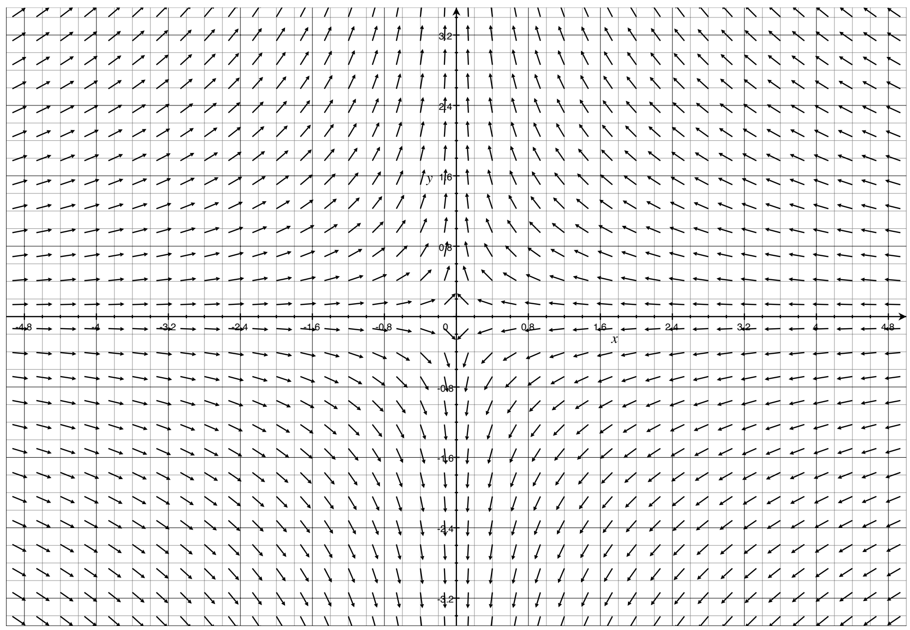
Thus the object will follow the path

$$l(t) = t(-b,-d) + (-a,-c)$$

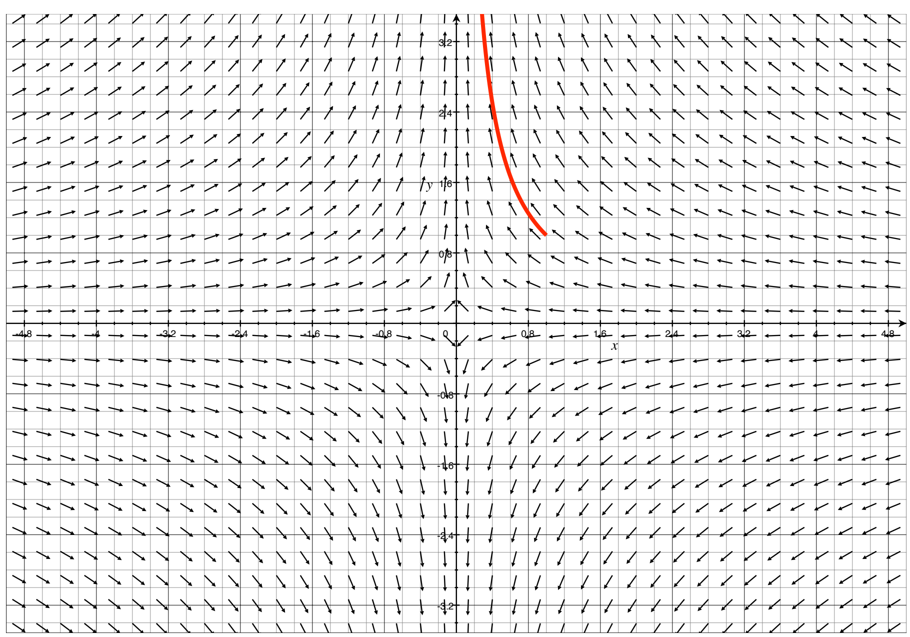
at time $t+\pi$ seconds. Hence, at time $t = \pi + 3$, the particle is at $l(3) = (-3b - a, -3d - c)$.

(14) Let $F(x, y) = (-x, y)$

(a) Sketch a portion of the vector field $F(x, y) = (-x, y)$.



(b) Sketch a flow line for the vector field starting at $(1, 1)$.



(c) Find a parameterization for the flow line starting at $(1, 1)$.

Solution: $\phi(t) = (e^{-t}, e^{t})$.

(d) The vector field F is a gradient field. Find the potential function.

Solution: $f(x, y) = -x^2/2 + y^2/2$.

(15) Let R be the region in $\mathbb{R}^2$ bounded by the lines $x = 0$, $y = x$ and $y = -x + 2$. Let $f(x, y) = x^2 + y^2$. Find $\iint\limits_R f\,dA$.

Solution: By Fubini's theorem:

$$\iint\limits_R f\,dA = \int_0^2 \int_x^{-x+2} x^2 + y^2\,dy\,dx.$$

(16) Let $f(x, y) = ye^x$. Find the gradient of f.

Solution: $\nabla f(x, y) = (ye^x, e^x)$.

(17) Let $F(x, y, 0) = (ye^x, xe^{y^2}, 0)$. Find the divergence of F.

Solution: $\operatorname{div} f(x, y) = ye^x + 2yxe^{y^2}$

(18) Let $F(x, y, z) = (xyz, xe^y \ln(z), x^2 + y^2 + z^2)$. Find the curl of F.

Solution:

$$\begin{pmatrix} 2y - xe^y/z \\ xy - 2x \\ e^y \ln z - xz \end{pmatrix}$$

(19) Find the curl of your answer to problem 16.

Solution:0

(20) Find the divergence of your answer to problem 18.

Solution: 0.