MA 302: HW 7 additional problems

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A: Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. This vector field is defined on $\mathbb{R}^2 - \{0\}$. In HW 6 you proved that $\operatorname{curl}\mathbf{F} = \mathbf{0}$. We know the following facts (all of which are 100% guaranteed!):

(1) $\mathbf{F}$ is not a gradient field on $\mathbb{R}^2 - \{0\}$. (That $\int_C \mathbf{F}\cdot d\mathbf{s} = 2\pi$ for a positively oriented simple closed curve enclosing the origin is one instance of this.)

(2) Let X^+ be the interval $[0,\infty)$ on the x-axis in $\mathbb{R}^2$. The set $\mathbb{R}^2 - X^+$ is simply connected. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a gradient field on $\mathbb{R}^2 - X^+$.

(3) Let X^- be the interval $(-\infty, 0]$ on the x-axis in $\mathbb{R}^2$. The set $\mathbb{R}^2 - X^-$ is simply connected. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a gradient field on $\mathbb{R}^2 - X^-$.

(4) $\mathbb{R}^2 - \{0\} = X^+ \cup X^-$.

Do these facts contradict each other? If not, why not?

Problem B: Suppose that $D \subset \mathbb{R}^2$ is a region and that $\mathbf{F}\colon D \to \mathbb{R}^2$ and $\mathbf{G}\colon D \to \mathbb{R}^2$ are C^1 vector fields. Suppose that for every oriented simple closed curve C in C, $\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C \mathbf{G}\cdot d\mathbf{s}$. Prove that there exists a scalar function $f\colon D \to \mathbb{R}^2$ such that $\mathbf{F} = \mathbf{G} + \nabla f$.

(Hint: Prove that the vector field $\mathbf{F} - \mathbf{G}$ has the path independence property by using Theorem 3.2 from the text.)

Problem C: (Extra-Credit) Use Problem B and the additional problems from HW 6 to prove the following:

Suppose that $\mathbf{F}\colon \mathbb{R}^2 - \{0\}$ is a C^1 vector field such that $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Then there exists a scalar $k \in \mathbb{R}$ and a scalar function $f\colon \mathbb{R}^2 - \{0\}$ such that for all (x,y):

$$\mathbf{F}(x,y) = \frac{k}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} + \nabla f.$$

Here is a way of rephrasing this result: Let V be the real vector space consisting of C^1 vector fields on $\mathbb{R}^2 - \{0\}$ with $\mathbf{0}$ curl. Let W be the real vector space consisting of C^1 gradient fields on $\mathbb{R}^2 - \{0\}$. Then the quotient vector space $H^1(\mathbb{R}^2 - \{0\}) = V/W$ is 1-dimensional.

Problem D: (even more extra-credit) Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be n distinct points in $\mathbb{R}^2$. Let $D = \mathbb{R}^2 - \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$. Prove that $H^1(D)$ is an n-dimensional vector space.

(If this interests you, you may be interested in the classroom scene in the movie “A Beautiful Mind”. You can see part of the scene here:

`http://www.haverford.edu/math/lbutler/MITclassroom.mov`

Pay attention to the problem on the board. Can you spot the difference between the problem stated in the movie and the version appearing here?

Read an article about the scene at:

`http://www.sciencemag.org/cgi/reprint/295/5556/789.pdf`)