Solutions to additional problems on HW 3

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A: Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a parameterized curve. In class, we learned how to reparameterized ϕ to become a curve $\phi(s)$ so that for $0 \le s \le t$, $\phi(s)$ has length t. Let $k > 0$ be a constant. Show how to reparameterize ϕ to a curve $\phi(u)$ so that for $0 \le u \le t$, $\phi(u)$ has length exactly kt.

Problem Solution:: Let $\mathbf{y}(t)$ be the result of reparameterizing $\phi(t)$ by arc length. Define $\psi(t) = \mathbf{y}(kt)$.

Claim: The arc length of ψ on the interval $[0,t]$ is kt.

proof of Claim: Notice that $\psi'(t) = k\mathbf{y}'(kt)$ by the Chain Rule. Thus,

$$\int_0^t ||\psi'(\tau)||\,d\tau = k\int_0^t ||\mathbf{y}'(k\tau)||\,d\tau.$$

Let $w = k\tau$. Then $dw = k\,d\tau$. Thus, by substitution the above expression is equal to

$$k\int_0^{kt} ||\mathbf{y}'(w)||(1/k)\,dw \quad = \quad \int_0^{kt} ||\mathbf{y}'(w)||\,dw.$$

Since $\mathbf{y}$ is parameterized by arc length, $\int_0^{kt} ||\mathbf{y}'(w)||\,dw = kt$. Thus, the arclength of ψ on $[0,t]$ is kt. □(Claim)

Recall that $\mathbf{y} = \phi \circ s^{-1}$ where $s(t) = \int_a^t ||\phi'(\tau)||\,d\tau$. Let $\alpha(t) = kt$. Then,

$$\psi = \mathbf{y} \circ \alpha = \mathbf{x} \circ s^{-1} \circ \alpha.$$

Notice that $s^{-1} \circ \alpha = (\alpha^{-1} \circ s)^{-1}$ so that

$$s^{-1} \circ \alpha(t) = (s(t)/k)^{-1}$$

That is, if

$$u(t) = \frac{1}{k}\int_a^t ||\phi'(\tau)||\,d\tau$$

then

$$\mathbf{x} \circ \mathbf{u}^{-1}$$

is a reparameterization of $\mathbf{x}$ with the desired property.

Problem B: Suppose that at time t a circle of radius ρ is tangent to the parameterized curve $\phi(t)$ where $\phi : \mathbb{R} \to \mathbb{R}^2$. What is a parameterization for the path taken by the center of the circle? (There are two possible answers, depending on which side of the curve the circle lie.)

Problem Solution:: At time t, the tangent vector to $\phi(t) = (x(t), y(t))$ is $\phi'(t) = (x'(t), y'(t))$. This tangent vector is perpendicular to both $\mathbf{v}(t) = (-y'(t), x'(t))$ and $\mathbf{w}(t) = (y'(t), -x'(t))$. Both vectors have magnitude, $||\phi'(t)||$. Thus the center of the circle lies at either $\mathbf{c}(t) = \phi(t) + \rho\mathbf{v}(t)/||\mathbf{v}(t)||$ or $\mathbf{c}(t) = \phi(t) + \rho\mathbf{w}(t)/||\mathbf{w}(t)||$.

Problem C: Suppose that at time $t = 0$, a circle of radius ρ is tangent to the parameterized curve $\phi(t)$. The circle rolls along the image of $\phi(t)$ in such a way that at time t, the circle is tangent to $\phi(t)$. Let P be the point on the circle directly to the right of the center of the circle at $t = 0$. In this problem, you will find a parameterization of the path $\mathbf{x}(t)$ taken by P.

(1) Find a parameterization $\mathbf{c}(t)$ for the path taken by the center of the circle. (Hint: use your work from problem B above.)

Problem Solution:: Use either of the formulas for $\mathbf{c}(t)$ given in the solution to Problem B.

(2) Find coordinates for $\mathbf{x}(t)$ in the tangent space $T_{\mathbf{c}}(t)$.

Problem Solution:: Let $s(t)$ denote the arclength of ϕ after t seconds. After t seconds, the circle has rolled a distance of $s(t)$ along the image of $\phi(t)$. (This is because the circle is always tangent to the image of ϕ at $\phi(t)$.) The distance the circle travels along the curve ϕ after t seconds is ρ times the angle $\theta(t)$ through which P travels (clockwise). (This is because we are measuring θ in radians.) Consequently,

$$s(t) = \rho\theta(t).$$

The rotation of the circle is parameterized as $\mathbf{x}(t) = (\cos(-\theta(t)), \sin(-\theta(t)))$. The negative sign was introduced because the circle is rotating clockwise. If it is rotating counter-clockwise you don't need the negative sign. (The problem doesn't actually specify which direction the circle is rolling.) Thus, in tangent space coordinates,

$$\mathbf{x}(t) = \rho(\cos(-s(t)/\rho), \sin(-s(t)/\rho)),$$

where $s(t) = \int_a^t ||\phi'(\tau)||\, d\tau$.

(3) Find coordinates for $\mathbf{x}(t)$ in the usual coordinate system.

Problem Solution:: $\mathbf{x}(t) = \mathbf{c}(t) + \rho(\cos(-s(t)/\rho), \sin(-s(t)/\rho))$.

Problem D: This is the same problem as problem C, but instead of the circle being tangent to the image of $\phi(t)$ at time t, the circle is rolling along the image of $\phi(t)$ so that it makes 1 revolution per second.

(1) At time t seconds, how far along the image of ϕ has the circle rolled?

Problem Solution:: $2\pi\rho t$.

(2) Find a reparameterization $\psi(t)$ of $\phi(t)$ so that at time t, the circle is tangent to the point $\psi(t)$. (Hint: Use problem A above)

Problem Solution:: Define $k = 2\pi\rho$ and use the parameterization $\psi(t)$ given by Problem A to get $\mathbf{c}(t)$.

(3) Find coordinates for $\mathbf{x}(t)$ in the usual coordinate system. (Your answer will likely have an inverse function in the expression – you won't be able to get it in closed form.)

Problem Solution:: Use $\mathbf{x}(t) = \mathbf{c}(t) + \rho(\cos(-s(t)/\rho), \sin(-s(t)/\rho))$.