MA 111 Probability Review

1. Some definitions, equations, and examples

A probability space $\mathcal{P}$ for an experiment (such as tossing a coin 4 times and recording the results) is the list of all possible outcomes from the experiment with a number associated to each. The numbers are the probabilities and they must be non-negative and must sum to 1. In frequentist probability the number corresponds to the fraction of occurences of the outcome out of a large number of repetitions of the experiment.

For example, if a single die is rolled 2 times, the list of all possible outcomes is

$$\mathcal{P} = \left\{ \begin{array}{c} 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, \\ 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, \\ 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66 \end{array} \right\}$$

In principle any non-negative numbers can be assigned to these outcomes so long as the sum is 1. If we are considering a fair die, however, we would assign a probability of 1/36 to each outcome as each outcome is equally likely.

In general if we have an experiment with two outcomes A and B and if we want to know the probability of the outcome "A then B" if the experiment is performed twice, the probability is

$$P(\text{“}A \text{ then } B\text{”}) = P(A)P(B)$$

so long as the outcomes are independent.

An event is merely a subset of $\mathcal{P}$. For example, "An odd number is rolled both times" is an event if the experiment is "roll a die twice and record the results". In this case, if E denotes the event,

$$E = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}$$

The probability of an event is the sum of the probabilities of the outcomes in the event. For example, in the previous situation

$$\begin{aligned} P(E) &= P(11) + P(13) + P(15) + P(31) + P(33) + P(35) + P(51) + P(53) + P(55) \\ &= 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/36 \\ &= 9/36 \end{aligned}$$

If the outcomes were not all equally likely you would simply use the correct probabilities in place of the 1/36.

If E and F are two events, then the probability of E given F is determined by the formula:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

where $E \cap F$ denotes the event where both E and F occur.

For example, suppose that a die is rolled twice and that the first number rolled is odd. What is the probability that the second number rolled is odd?

Answer: Let F be the event that the first number rolled is odd. Let E be the event that the second number rolled is odd. We wish to know $P(E|F)$. The event $E \cap F$ is the event where both numbers rolled are odd. We determined previously that $P(E \cap F) = 9/36$. It should be easy to figure out that $P(F) = 1/2$. Thus, $P(E|F) = (9/36)/(1/2) = (1/4)/(1/2) = 1/2$

Sometimes, it is impractical to list all the possible outcomes for an experiment or event. In this case, the notion of "choose" is sometimes useful. The phrase "n choose k" means "the number of ways to choose k objects out of a collection of n objects".

For example, what is the probability that if a fair die is rolled 10 times, exactly 2 of the times an odd number appears?

Answer 1: Consider writing the results of the 10 rolls into the 10 blanks below:

$$\begin{array}{ccccc} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{array}$$

We must choose 2 of these blanks to contain an odd number. All the rest must contain an even number. There are "10 choose 2" = 45 ways of choosing the two rolls where the odd number will occur. There are 3 choices of odd number for each of those places. Thus there are $3^2 \cdot 45 = 405$ ways of choosing two spots and putting an odd number in each spot. The other 8 spots all need to have an even number put in them. Since there are 3 even numbers, there are 3^8 ways of putting even numbers into the remaining 8 spots. Thus the total number of ways of rolling exactly two odd numbers is

$$3^8 \cdot 3^2 \cdot 45 = 3^{10} \cdot 45 = 2657205.$$

The number of total possible outcomes of the experiment of rolling a die 10 times is $6^{10} = 3^{10} \cdot 2^{10}$. Thus, the probability of rolling exactly 2 odd numbers out of 10 rolls is:

$$\frac{3^{10} \cdot 45}{3^{10} \cdot 2^{10}} = \frac{45}{2^{10}} \approx .044$$

Answer 2: Here is another way of doing this. Let F be the event that there exactly 2 out 10 rolls are odd numbers. An example of an outcome in F is

$$EEOEOEEEEE$$

The probability of rolling an even is $1/2$. The probability of rolling an odd is $1/2$. Thus,

$$P(EEOEOEEEEE) = \frac{1}{2^{10}}.$$

Since every outcome in F has exactly 2 odds and 8 evens, the probability of every outcome in F is $1/2^{10}$. The probability of F is the sum of the probabilites of the outcomes in F, thus

$$P(F) = 1/2^{10} + \ldots + 1/2^{10} = (\text{\# of outcomes in } F) \cdot (1/2^{10}).$$

How many outcomes are there in F? There are exactly "10 choose 2" = 45 ways of choosing exactly two rolls to be odd. Thus $P(F) = 45/2^{10}$ as before.

Notice: If the probabilities of rolling an even and an odd were not equal, the method of Answer 2 would still work while the method of Answer 1 would not. You might try answering the question if the probability of rolling an even is $3/5$ whilst the probability of rolling an odd is $2/5$.

2. Some other things to be able to do

(Note: The problems will be similar to examples done in class, problems on Problem Sets, and examples on this review sheet. Also, you do not need a calculator for the exam; you will be allowed to leave numbers in an uncalculated form. The problems should always be worked to the point where all that is left to be done is to plug the numbers into a calculator.)

(1) Calculate the probability that somebody wins an unfinished game.
(2) Be able to summarize Pascal's difficulties with Fermat's solution to the unfinished game. Be able to summarize how Pascal set about trying to understand Fermat's solution.
(3) Calculate the probability of an event.
(4) If event B is independent of event A, be able to calculate the probability of the event "A then B".
(5) Calculate the probability of an event given additional information.
(6) Calculate average wins or losses if a betting game is played many times.
(7) Know the definitions of frequentist and subjective probabilities.
(8) Know who Pascal, Fermat, Cardano, and Graunt were. (See Devlin's book)

(9) Know the definition of expected value (expectation) and be able to calculate it.
(10) Be able to figure out the outcomes in a given event
(11) Be able to explain the calculations involved in the birthday paradox.
(12) Understand and be able to explain the prosecutor's fallacy.
(13) Given two conditional probabilities, be able to figure out which is relevant to a particular situation. Given a situation, be able to figure out which conditional probability is relevant.