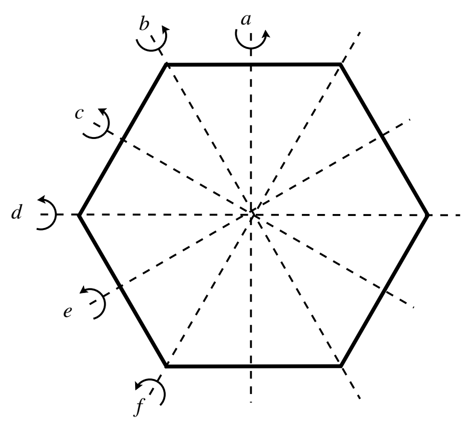
Problem Set 5

MA 111 Spring 2010

Give complete and thorough answers to these problems on separate sheets of paper. The assignment is due in class on April 2. This assignment is for extra-credit.

Problem A: Do the symmetries $[1 \leftrightarrow 2] \circ [3 \leftrightarrow 4]$ and $[2 \to 3 \to 4 \to]$ generate $\mathbb{S}_4$? Why or why not? If they do not, how many elements are in the subgroup generated by them?

Problem B: Recall that D_6 consists of the symmetries of a regular hexagon. Call the upper right corner of the hexagon P. Label the reflections as in the picture. The rotational symmetries are $\mathbf{I}$, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}.



(1) Write down all the symmetries in D_6 which do not move P. Explain why this is a subgroup of D_6. Call this subgroup H. (It is called the "stabilizer" of P.)

(2) For each corner of the hexagon, write down all the symmetries in D_6 which move P to that corner. Show that these are the cosets of H in D_6.

Problem C: Let X be an object and let G be its group of symmetries. Assume that G has finitely many symmetries. Let p be a specific point in X. (You may wish to compare this problem to the previous one.)

(1) Let H be the set of all symmetries in G which do not move P. Explain why H is a subgroup of G. (It is called the "stabilizer" of x in G.)

(2) Suppose that S and T are two symmetries in G which move P to the same point Q. Prove that S and T are in the same coset of H in G. (In other words, you must show that there exists a symmetry h in H so that $S = T \circ h$.

(3) Suppose that S and T are two symmetries in G which are in the same coset of H in G. Prove that S and T move P to the same point in X.

(4) If P can be moved to a point Q in X by a symmetry in G, we say that Q is in the "orbit" of P. Use the previous parts of this problem and LaGrange's theorem to show that the number of symmetries in the stabilizer of P times the number of points in the orbit of X is equal to the number of symmetries in G.

Problem D: Show that the symmetries $[1\,2\,3]$, $[1\,2\,4]$, $[1\,3\,4]$, $[2\,3\,4]$ generate $\mathbb{A}_4$. Use this to show that the symmetries of a tetrahedron are the group $\mathbb{A}_4$.