Sperner's Lemma

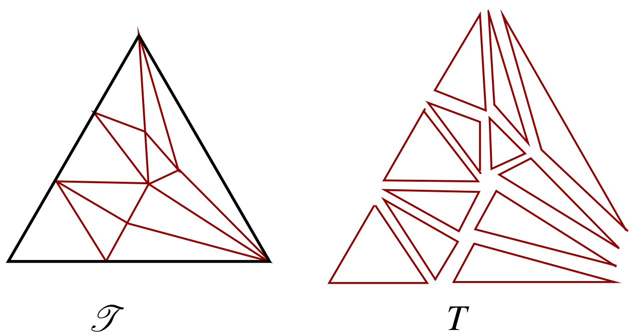
The purpose of this note is to elaborate the proof of Sperner's Lemma.

Theorem (Theorem 2.25)**.** Let $\mathscr{T}$ be a triangulation of an n–dimensional simplex Δ. Suppose that each vertex of $\mathscr{T}$ has been assigned a number from the set $\{0,\ldots,n\}$. Then the number of completely labelled n–simplices of $\mathscr{T}$ is odd if and only if the number of completely labelled $(n-1)$–simplices of the triangulation $\mathscr{T} \cap \partial\Delta$ is odd.

The first step of Theorem 2.25 is: Suppose that D is an n–simplex of $\mathscr{T}$. Then the following statments hold:

(1) D is completely labelled $\Leftrightarrow$ D has exactly exactly one completely labelled $(n-1)$–dimensional face.
(2) D is not completely labelled $\Leftrightarrow$ either D has no completely labelled faces or D has two completely labelled faces.

Pull apart $\mathscr{T}$ to obtain a collection T of disjoint n–simplices. (See the figure.)



From (1) and (2) we obtain:

(3) The number of completely labelled n–simplices in $\mathscr{T}$ is congruent modulo 2 to the number of completely labelled $(n-1)$–dimensional faces of T.

Now it is clear that:

(4) $\{$ completely labelled $(n-1)$–faces of $T\} =$
$\{$ completely labelled $(n-1)$ faces of T on $\partial\Delta\} \cup$
$\{$ completely labelled $(n-1)$–faces of T not on $\partial\Delta\}$

Notice that the two sets on the right hand side are disjoint.

Consider an $(n-1)$–face F of T. If F appears on the boundary of Δ then it is in exactly one n simplex of T. If F is not on $\partial\Delta$ then F appears in exactly two n–simplices of T. Thus, the set

$$\{ \text{ completely labelled } (n-1)\text{–faces of } T \text{ not on } \partial\Delta\}$$

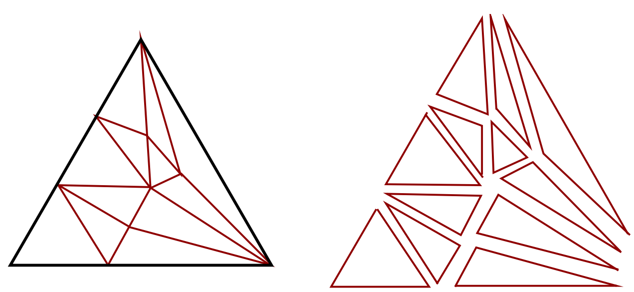
has an even number of elements. That is

(5) The number of completely labelled $(n-1)$–dimensional faces of T is congruent modulo 2 to the number of completely labelled $(n-1)$–faces on $\partial\Delta$.

Putting observations (3) and (5) together we obtain

- The number of completely labelled n–simplices in $\mathscr{T}$ is congruent modulo 2 to the number of completely labelled $(n-1)$–faces on $\partial\Delta$.

This is what we were to prove.



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