

## Simplices

**Definition 1.** Suppose that  $v_0, \dots, v_k \in \mathbb{R}^n$ . The **convex hull** of  $\{v_0, \dots, v_k\}$  is the smallest convex set containing  $v_0, \dots, v_k$ . It is denoted  $CH(v_0, \dots, v_k)$ . It turns out that

$$CH(v_0, \dots, v_k) = \left\{ w \in \mathbb{R}^n : \exists \lambda_0, \dots, \lambda_k \in \mathbb{R} \text{ s.t. } w = \sum \lambda_i v_i \text{ and } \sum \lambda_i = 1 \right\}.$$

**Definition 2.** A **vector subspace** of  $\mathbb{R}^n$  is a subset which is closed under (finite) linear combinations. An **affine subspace** is a subset which is the translation of a vector subspace. That is,  $W$  is an affine subspace of  $\mathbb{R}^n$  if and only if there exists a vector subspace  $V$  and a vector  $a$  such that for each  $w \in W$  there exists  $v \in V$  so that  $w = a + v$ . We denote this

$$W = a + V.$$

The dimension of  $W$  is defined to be the dimension of  $V$  (as a subspace of  $\mathbb{R}^n$  – this is not topological dimension).

**Definition 3.** Suppose that  $v_0, \dots, v_k \in \mathbb{R}^n$ . Then  $v_0, \dots, v_k$  are **affinely independent** iff for each collection of  $m + 1$  distinct points  $w_0, \dots, w_m \in \{v_0, \dots, v_k\}$  there is no  $m - 1$  dimensional affine subspace containing  $w_0, \dots, w_m$ . Equivalently,  $v_0, \dots, v_k$  are affinely independent if the vectors  $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$  are linearly independent in  $\mathbb{R}^n$ .

**Example.** Two points are affinely independent if and only if they are not the same. Three points are affinely independent if and only if they are not collinear.  $\mathbb{R}^n$  contains at most  $n + 1$  affinely independent points.

**Definition 4.** A  **$k$ -dimensional simplex**  $\Delta$  in  $\mathbb{R}^n$  is the convex hull of  $k + 1$  affinely independent points. If we need to specify the points, we will sometimes write  $\Delta = \Delta(v_0, \dots, v_k)$ , where  $v_0, \dots, v_k$  are the affinely independent points. An  $l$ -dimensional **face** of  $\Delta$  is the convex hull of distinct points  $w_0, \dots, w_l \in \{v_0, \dots, v_k\}$ . We consider  $\emptyset$  to be a  $(-1)$ -dimensional face of every simplex. The **standard  $n$ -simplex** is the convex hull of  $0, e_1, \dots, e_n$  where  $e_i$  is the  $i$ th standard basis vector of  $\mathbb{R}^n$ .

**Definition 5.** Suppose that  $P \subset \mathbb{R}^n$  is the union of finitely many simplices  $\mathcal{T}$  (not necessarily of the same dimension). Then  $\mathcal{T}$  is a (geometric) **triangulation** of  $P$  if whenever  $\sigma, \tau$  are simplices in  $\mathcal{T}$  then  $\sigma \cap \tau$  is a face of each.