Closed Sets and Sequential Compactness

Definition 1. Let X be a topological space and suppose that $\{x_n\}$ is a sequence in X and that $l \in X$. Then l is a **limit point** of the sequence $\{x_n\}$ iff for every open set U containing l there is an $N \in \mathbb{N}$ so that for all $n \geq N$, $x_n \in U$.

Suppose that $A \subset X$ and that $l \in X$. Then l is a **limit point** of A iff for every open set U containing l, there is an $a \in U \cap A$ such that $a \neq l$. The **closure** of A, denoted $\overline{A}$ is defined to be $A \cup$ limit points of$A$.

Example. Consider the sequence $\{x_n\} \subset \mathbb{R}$ where, for all n, $x_n = 0$. Then 0 is a limit point of $\{x_n\}$ as a sequence, but not as a set. In other words, the two definitions of limit point do not necessarily agree for sequences.

Lemma 2. Suppose that X is a topological space and that $A \subset X$. Then $\overline{A}$ is a closed subset of X.

Proof. We will show that $X \backslash \overline{A}$ is an open set. Let $c \in X \backslash \overline{A}$. Since c is neither in A nor is a limit point of A, there exists an open set U containing c so that $U \cap A = \varnothing$. Suppose that $l \in U$. Since U is an open set containing l and is disjoint from A, l cannot be a limit point of A. Also, $l \notin A$. Thus, $\overline{A} \cap U = \varnothing$. Thus, U is an open set containing c which is disjoint from $\overline{A}$. Since c was arbitrary, $X \backslash \overline{A}$ is open. □

Lemma 3. Suppose that X is a topological space and that $A \subset X$. Suppose that C is a closed set containing A. Then $\overline{A} \subset C$.

Proof. Suppose that $l \in X \backslash C$ is a limit point of A. Since C is closed, $X \backslash C$ is open. Since l is a limit point of a, the set $A \cap X \backslash C$ is non-empty. But this contradicts the hypothesis that $A \subset C$. □

The previous two sets show that $\overline{A}$ is the smallest closed set in X containing A. This observation proves that:

Corollary 4. Suppose that $A \subset X$. Then A is closed if and only if $A = \overline{A}$.

An immediate application is:

Corollary 5. Suppose that $A \subset \mathbb{R}$ is a closed, bounded set. Then $\inf A \subset A$ and $\sup A \subset A$.

Proof. Notice that $\inf A$ and $\sup A$ exist as A is a bounded subset of $\mathbb{R}$. It suffices to show that $\inf A$ and $\sup A$ are limit points of A. We will do this for $\sup A$. By definition of supremum, if $\alpha \geq x$ for all $x \in A$, then $\alpha \geq \sup A$. Thus, each interval of the form $(\sup A - \varepsilon, \sup A]$ contains a point of A. Since every open set containing $\sup A$ contains an interval of that form, $\sup A$ is a limit point of A. $\square$

Here is another application:

Corollary 6. The topological dimension of $[0,1]$ is at least 1.

Proof. We must show that there exists $\varepsilon > 0$ such that for all finite closed covers $\mathscr{U}$ of $[0,1]$ so that for all $U \in \mathscr{U}$, $\operatorname{diam} U < \varepsilon$, there exist two distinct sets $U_1, U_2 \in \mathscr{U}$ such that $U_1 \cap U_2 \neq \varnothing$. (i.e. the order of $\mathscr{U}$ is at least two.)

Choose $\varepsilon = 1/2$ and let $\mathscr{U}$ be a finite closed cover of $[0,1]$ so that each set in $\mathscr{U}$ has diameter less than $\varepsilon = 1/2$. Let U_1 be a set in $\mathscr{U}$ containing 0. Since $\operatorname{diam}[0,1] = 1$, $1 \notin U_1$. Let $\alpha = \inf U_1$. Since U_1 is closed, $\alpha \in U_1$. Consider the points $x_n = \alpha + 1/n$ (for which $\alpha + 1/n < 1$). The cover $\mathscr{U}$ is finite, so there exists $U \in \mathscr{U}$ which contains infinitely many of the x_n. Since the sequence $\{x_n\}$ converges to α, α is a limit point of U. (proof?) Since U is closed, $\alpha \in U$. Thus $\alpha \in U \cap U_1 \neq \varnothing$. $\square$

Definition 7. A topological space X is **sequentially compact** iff every sequence $\{x_n\} \subset X$ has a convergent subsequence.

Theorem 8. If a metric space (X,d) is compact then it is sequentially compact.

In fact, the converse also holds, but the proof is more difficult.

Proof. Suppose that X is a compact metric space. Let $S = \{x_n\}$ be a sequence in X. We wish to show that S has a convergent subsequence.

Case 1: S is a closed subset of X.

Since X is a compact metric space, S being closed implies that S is compact. Let

$$\varepsilon_n = \min\{d(x_n, x_m) | m < n \text{ and } x_m \neq x_n\}.$$

Notice that ε_n exists and is nonzero. Then $\mathscr{B} = \{B_{\varepsilon_n}(x_n)\}$ is an open cover of S such that distinct points of S are in disjoint sets in the cover. Since S is compact, there are a finite number of points $x_1, \ldots, x_n$ so that $\{B_{x_i} | 1 \leq i \leq n\}$ is a cover of S. In other words, as a set $S = \{x_1, \ldots x_m\}$. Since S is an infinite sequence, there is some point x_k which appears infinitely often. Let $\mathscr{N} = \{n \in \mathbb{N} : x_n = x_k\}$. Then $\{x_n : n \in \mathscr{N}\}$ is a subsequence of $\{x_i\}$ which is constant, and therefore converges.

Case 2: S is not closed.

Since S is not closed, $S \neq \overline{S}$. Let $l \in \overline{S}$. If $m \in \mathbb{N}$, the set $B_{1/m}(l) \cap S$ is non-empty (since l is a limit point of S). Let x_m be a point in $B_{1/m}(l) \cap S$. These points $\{x_m\}$ are a subsequence of $\{x_n\}$ which converges to l. $\square$