Point-Set Topology

1. Topological Spaces and Continuous Functions

Definition 1.1. Let X be a set and $\mathcal{T}$ a subset of the power set $\mathcal{P}(X)$ of X. Then $\mathcal{T}$ is a **topology** on X if and only if all of the following hold

(a) $\varnothing \in \mathcal{T}$
(b) $X \in \mathcal{T}$
(c) (Arbitrary unions) If $A_\alpha \in \mathcal{T}$ for α in some index set I then

$$\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}.$$

(d) (Finite intersections) If $B_\beta \in \mathcal{T}$ for β in some finite set I then

$$\bigcap_{\beta \in I} B_\beta \in \mathcal{T}.$$

If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is a **topological space**. Sometimes, if $\mathcal{T}$ is understood we say that X is a topological space. If $U \in \mathcal{T}$ then U is an **open** set. If $V \subset X$ and $X \backslash V$ is open, then V is **closed**. Notice that in any topological space $(X, \mathcal{T})$, both X and $\varnothing$ are sets which are both open and closed. It is possible for a set to be neither open nor closed.

Exercise 1.2. Show that the Finite intersections axiom is equivalent to:

- If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.

Exercise 1.3. Let X be any set. Prove that the following are topologies on X

(a) (the discrete topology) $\mathcal{T} = \mathcal{P}(X)$
(b) (the indiscrete topology) $\mathcal{T} = \{\varnothing, X\}$

Exercise 1.4. Let $X = \{a, b, c, d\}$. Find all topologies on X containing five or fewer sets and prove that your list is complete.

Exercise 1.5. Let $(X, \mathcal{X})$ be a topological space. Prove that a set $U \subset X$ is open if and only if for all $x \in U$ there exists an open set U_x such that $x \in U_x$ and $U_x \subset U$.

Definition 1.6. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. Let $\mathcal{T}$ be the smallest topology on X containing $\mathcal{B}$. That is, $\mathcal{B} \subset \mathcal{T}$ and if $\mathcal{T}'$ is a topology on X such that $\mathcal{B} \subset \mathcal{T}'$ then $\mathcal{T} \subset \mathcal{T}'$. We say that $\mathcal{B}$ **generates** the topology $\mathcal{T}$.

Exercise 1.7. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. Consider the set $\mathbb{T}$ consisting of all topologies on X such that each set in $\mathcal{B}$ is open. Let $\mathcal{T}$ be the intersection of all the topologies in $\mathbb{T}$. Prove that $\mathcal{T}$ is a topology on X and that $\mathcal{B}$ generates $\mathcal{T}$.

Exercise 1.8. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. Let $\mathcal{T}$ denote the set which contains $U \subset X$ if and only if at least one of the following holds:

- U is the (arbitrary) union of sets in $\mathcal{B}$
- U is the finite intersection of sets in $\mathcal{B}$
- $U = \varnothing$
- $U = X$.

Prove that $\mathcal{T}$ is a topology on X and that $\mathcal{B}$ generates $\mathcal{T}$.

Definition 1.9. Let $(X, \mathcal{T})$ be a topological space. A set $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ if each element of $\mathcal{T}$ can be written as the union of elements of $\mathcal{B}$.

Remark 1.10. Notice that a generating set $\mathcal{B}$ for a topology may or may not be a base for the topology.

Exercise 1.11. Let $X = \mathbb{R}$ and let $\mathcal{B}$ consist of all intervals in $\mathcal{R}$ of the form (a, b) with $a < b$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. $\mathcal{T}$ is called the **usual** topology on $\mathbb{R}$. You do not need to give completely rigourous answers to the following questions.

(a) Describe the sets in $\mathcal{T}$.
(b) Give an example of a set in $\mathbb{R}$ which is neither open nor closed.

Definition 1.12. Let X be a set and let $d\colon\ X \times X \to \mathbb{R}$ be a function such that

(a) (Positive) For all $x, y \in X$, $d(x, y) \geq 0$
(b) (Definite) $d(x, y) = 0$ if and only if $x = y$.
(c) (Symmetric) For all x, y, $d(x, y) = d(y, x)$.
(d) (Triangle inequality) For all $x, y, z \in \mathbb{R}$, $d(x, z) \leq d(x, y) + d(y, z)$

The function d is a **metric** on X and (X, d) is a metric space.

Exercise 1.13. Prove that the following are metrics on $\mathbb{R}$.

(a) (the usual metric) $d(x, y) = |x - y|$
(b) (the discrete metric) $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

Exercise 1.14. Prove that the following are metrics on $\mathbb{R}^n$. Let x_i denote the ith component of $x \in \mathbb{R}^n$.

(a) (the usual metric) $d^u(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$
(b) (the sup metric) $d^s(x,y) = \max_i\{|x_i - y_i|\}$
(c) (the taxicab metric) $d^t(x,y) = \sum_i |x_i - y_i|$

Exercise 1.15. (The comb metric) Let $X = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$. Consider the following function $d\colon X \times X \to \mathbb{R}$:

$$d((x_1,x_2),(y_1,y_2)) = \begin{cases} 0 & \text{if } (x_1,x_2) = (y_1,y_2) \\ |x_2 - y_2| & \text{if } x_1 = y_1 \\ |x_2| + |y_2| + |x_1 - y_1| & \text{if } x_1 \neq y_1 \end{cases}$$

Prove that d is a metric and describe the shortest path between two points in X.

Definition 1.16. Let (X,d) be a metric space. For $x \in X$ and $\epsilon > 0$ let

$$B_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}.$$

Let $\mathcal{T}$ be the topology generated by

$$\{B_\epsilon(x) : x \in X \text{ and } \epsilon > 0\}.$$

We say that d **generates** the topology $\mathcal{T}$.

Exercise 1.17. Let $\mathcal{T}$ be the topology generated by d. Show that the metric balls $B_\epsilon(x)$ are a base for $\mathcal{T}$.

Exercise 1.18. Prove that the usual metric on $\mathbb{R}$ generates the usual topology on $\mathbb{R}$.

Example 1.19. Prove that the topology generated by the usual metric is the same as the topology generated by the taxicab metric.

Proof. Let $\mathcal{T}^u$ and $\mathcal{T}^t$ denote the topologies generated by the usual metric and the taxicab metric respectively. We desire to show that $\mathcal{T}^u = \mathcal{T}^s$. Let $B^u_\epsilon(x)$ denote a ball in the usual metric and let $B^t(x)$ denote a ball in the taxicab metric. Since $\mathcal{T}^u$ is the smallest topology on $\mathbb{R}^n$ containing $\{B^u_\epsilon(x)\}$, if each $B^u_\epsilon(x)$ is in $\mathcal{T}^t$ then, since $\mathcal{T}^t$ is a topology, $\mathcal{T}^u \subset \mathcal{T}^t$. Similarly, if each $B^t_\epsilon(x)$ is in $\mathcal{T}^u$, then $\mathcal{T}^t \subset \mathcal{T}^u$. Thus, to show that $\mathcal{T}^t = \mathcal{T}^u$, we need only show that open balls in one metric space are open in the other metric space. Note, however, that they will likely not be open *balls* in the other metric space.

We begin by showing that each ball $B^u_\epsilon(x)$ is the union of open balls in the taxicab metric. By Exercise 1.5, this is equivalent to showing that for all $z \in B^u_\epsilon(x)$, there exists $\delta > 0$ so that $B^t_\delta(z) \subset B^u_\epsilon(x)$.

Let $\Delta = d(x, z)$. Choose δ so that $0 < \delta < \epsilon - \Delta$. Let $y \in B^t_\delta(z)$. We must show that $y \in B_\epsilon(x)$. This is equivalent to showing that $d(x, y) < \epsilon$.

By the triangle inequality,

$$d(x, y) \leq \Delta + d(z, y) = \Delta + \sqrt{\sum_i (z_i - y_i)^2}.$$

By properties of the square root:

$$\Delta + \sqrt{\sum_i (z_i - y_i)^2} \leq \Delta + \sum_i \sqrt{(z_i - y_i)^2} = \Delta + \sum_i |z_i - y_i|.$$

Notice that this means:

$$d(x, y) \leq \Delta + d^t(z, y) < \Delta + \epsilon - \Delta = \epsilon$$

as desired. This shows that $\mathcal{T}^u \subset \mathcal{T}^t$.

We next show that $\mathcal{T}^t \subset \mathcal{T}^u$. Let $z \in B^t_\epsilon(x)$. We desire to show that there exists $\delta > 0$ so that $B^u_\delta(z) \subset B^t_\epsilon(x)$. Before beginning the proof we need:

Lemma. For all non-negative real numbers a and b

$$\sqrt{a} + \sqrt{b} \leq 2\sqrt{a + b}.$$

Proof of Lemma. Square both sides of the inequality to obtain the equivalent inequality:

$$a + 2\sqrt{ab} + b \leq 4a + 4b.$$

Rearrange to obtain the equivalent inequality:

$$2\sqrt{ab} \leq 3a + 3b.$$

Square both sides to obtain the equivalent inequality

$$4ab \leq 9a^2 + 18ab + 9b^2.$$

Hence,

$$0 \leq 9a^2 + 14ab + 9b^2$$

is an inequality equivalent to the one we are trying to achieve. This is obviously true (since $a, b \geq 0$) so the original inequality is also true. $\square$

Let $\Delta = d^t(x, z)$. Choose δ so that $0 < \delta < (\epsilon - \Delta)/2^{n-1}$. Let $y \in B_\delta(z)$. We need to show that $d^t(x, y) < \epsilon$. By the triangle inequality:

$$d^t(x, y) \leq \Delta + d^t(z, y) = \Delta + \sum_{i=1}^{n} |z_i - y_i|.$$

Thus,

$$d^t(x,y) \leq \Delta + \sum_{i=1}^{n} \sqrt{(z_i - y_i)^2}$$

Combining the lemma with a proof by induction shows:

$$\sum_{i=1}^{n} \sqrt{(z_i - y_i)^2} \leq 2^{n-1} \sqrt{\sum_{i=1}^{n} (z_i - y_i)^2}.$$

Thus,

$$d^t(x,y) \leq \Delta + 2^{n-1} d(z,y).$$

Since $y \in B_\delta(z)$, by the definition of δ we have

$$d^t(x,y) < \Delta + 2^{n-1}\Big(\frac{\epsilon - \Delta}{2^{n-1}}\Big) = \Delta + \epsilon - \Delta = \epsilon.$$

Since y was arbitrary, $B^t_\epsilon(x) \in \mathcal{T}^u$. Thus, $\mathcal{T}^t \subset \mathcal{T}^u$.

Consequently, $\mathcal{T}^t = \mathcal{T}^u$ as desired. □

Exercise 1.20. Prove that the usual metric, the sup metric, and the taxicab metric all generate the same topology of $\mathbb{R}^n$.

Definition 1.21. Suppose that $f\colon X \to Y$ is a function and that $A \subset X$. The function $f|_A\colon A \to Y$ defined by $f|_A(a) = f(a)$ for all $a \in A$ is called the **restriction** of f to A. Often we will write f rather than $f|_A$.

Exercise 1.22. The usual metric on $\mathbb{R}^2$ restricts to be a metric on the closed upper half space of $\mathbb{R}^2$ (i.e. $\{(x,y) : y \geq 0\}$). Does the usual metric on the closed upper half space generate the same topology as the comb metric? (See Exercise 1.15.)

Definition 1.23. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Let $f\colon X \to Y$ be a function. If $U \subset Y$, let

$$f^{-1}(U) = \{x \in X : f(x) \in U\}.$$

Define f to be **continuous** if and only if for every open set $U \subset Y$ the set $f^{-1}(U)$ is open in X.

Exercise 1.24. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces and let $f\colon X \to Y$ be a function.

(a) Suppose that there exists $c \in Y$ such that $f(x) = c$ for all $x \in X$. (That is, f is a constant function.) Prove that f is continuous.
(b) Suppose that $\mathcal{X}$ is the discrete topology. Prove that f is continuous.
(c) Suppose that $\mathcal{Y}$ is the indiscrete topology. Prove that f is continuous.

Exercise 1.25. Let X and Y be topological spaces. Let $f\colon X \to Y$ be a function and let $\mathcal{B}$ be a base for a topology on Y. Suppose that for all $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X. Prove that f is continuous.

Exercise 1.26. Let (X, d) be a metric space. Give X the topology generated by d and give $\mathbb{R}$ the topology generated by the usual metric. Let $x_0 \in X$ and define $f\colon X \to \mathbb{R}$ by

$$f(x) = d(x_0, x).$$

Prove that f is continuous.

The next exercise connects the topological notion of continuity with the notion of continuity you learned about in Calculus classes.

Exercise 1.27. In analysis courses a function $f\colon \mathbb{R} \to \mathbb{R}$ is defined to be continuous at $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. The function is continuous if it is continuous at every point $x_0 \in \mathbb{R}$.

Give $\mathbb{R}$ the usual topology. Prove that f is continuous in the analysis sense if and only if it is continuous in the sense of Definition 1.23.

Exercise 1.28. (The composition of two continuous functions is continuous) Let $(X, \mathcal{X})$, $(Y, \mathcal{Y})$, and $(Z, \mathcal{Z})$ be topological spaces. Let $f\colon X \to Y$ be surjective and continuous. Let $g\colon Y \to Z$ be continuous. Prove that $g \circ f$ is continuous.

Definition 1.29. Let X and Y be sets. A function $f\colon X \to Y$ is

(a) **injective** if for every $x_1 \neq x_2 \in X$ $f(x_1) \neq f(x_2)$
(b) **surjective** if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
(c) **bijective** if f is injective and surjective

Recall that if $f\colon X \to Y$ is bijective there exists an inverse function

$$f^{-1}\colon Y \to X.$$

Definition 1.30. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. A function $f\colon X \to Y$ is a **homeomorphism** if f is bijective and both f and f^{-1} are continuous. We say that X and Y are **homeomorphic** if there exists a homeomorphism between them.

Exercise 1.31. Suppose that $a, b \in \mathbb{R}$ and $a < b$. The metric on $\mathbb{R}$ restricts to a metric on the intervals (a, b) and $(-\pi/2, \pi/2)$ giving the usual topologies on those intervals. Show that the open interval (a, b) (with the usual topology) is homeomorphic to the open interval $(-\pi/2, \pi/2)$ (with the usual topology).

Exercise 1.32. Give $\mathbb{R}$ and $(-\pi/2, \pi/2)$ the usual topologies. Explain why

$$\arctan\colon\ \mathbb{R} \to (-\pi/2, \pi/2)$$

is a homeomorphism. You may use Exercise 1.27 and well-known facts from Calculus.

Exercise 1.33. Find all sets X such that X with the discrete topology is homeomorphic to X with the indiscrete topology.

2. Constructing Topological Spaces

2.1. **The subspace topology.**

Definition 2.1. Let $(X, \mathcal{X})$ be a topological space and let $Y \subset X$. Define

$$\mathcal{Y} = \{U \cap Y : U \in \mathcal{X}\}.$$

$\mathcal{Y}$ is called the **subspace topology** on Y.

Exercise 2.2. Prove that the subspace topology is a topology.

Exercise 2.3. Let (X, d) be a metric space and let $Y \subset X$. Notice that $d|_Y$ is a metric on Y. Show that $d|_Y$ generates the subspace topology on Y.

2.2. **The product topology.**

Definition 2.4. Let X and Y be sets. The **product** $X \times Y$ is defined as the set of all ordered pairs (x, y) such that $x \in X$ and $Y \in Y$. For sets $X_1, X_2, \ldots, X_n$ the product $\times X_i$ is defined to be the set of n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$. For a countable collection of sets X_i for $i \in \mathbb{N}$, the product $\times X_i$ is defined to be the set of sequences $(x_1, x_2, \ldots)$ with $x_i \in X_i$.

Definition 2.5. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Let $Z = X \times Y$. Let

$$\mathcal{B} = \{U \times V : U \in \mathcal{X} \text{ and } V \in \mathcal{Y}\}.$$

The **product topology** on $\mathbb{Z}$ is the topology generated by $\mathcal{B}$. We will sometimes refer to the sets in $\mathcal{B}$ as **product open sets**.

Exercise 2.6. Explain why $\mathcal{B}$ in the definition above is usually not itself a topology on Z.

Exercise 2.7. Prove that the set of product open sets is a base for the product topology. Use this to show that if $f\colon\ Z \to X \times Y$ is a function between topological spaces, with $X \times Y$ having the product topology, then f is continuous if and only if the inverse image of every product open set is open.

Exercise 2.8. Let $(X, \mathcal{X})$, $(Y, \mathcal{Y})$ be topological spaces. Give $X \times Y$ and $Y \times X$ the product topologies. Prove that $X \times Y$ is homeomorphic to $Y \times X$.

Exercise 2.9. Show that the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the topology generated by the usual metric. (Hint: Use Exercise 1.20.)

Exercise 2.10. Let X and Y be topological spaces and define a function $\mathrm{pr}_X\colon X \times Y \to X$ by $\mathrm{pr}_X(x, y) = x$. Define pr_Y similarly. Show that the product topology on $X \times Y$ is the smallest topology on $X \times Y$ for which pr_X and pr_Y are continuous. The functions pr_X and pr_Y are called **projections**.

Definition 2.11. Let A be a finite or countably infinite set and let $(X_a, \mathcal{X}_a)$ be a topological space for each $a \in A$. Let $Z = \times_{a \in A} X_a$. Let

$$\mathcal{B} = \{\times_{a \in A} U_a : U_a \in \mathcal{X}_a \text{ and for all but finitely many } a, U_a = X_a\}.$$

The **product topology** on Z is the topology generated by $\mathcal{B}$. In fact, this definition makes sense for sets A which are uncountable, but we will never need that notion.

Exercise 2.12. Use the same notation as in Definition 2.11. Prove that the product topology on Z is the smallest topology for which the projections are continuous. (The definition of projection in this context should be obvious.) If you like, you may simply describe how to adapt your proof from Exercise 2.10.

Exercise 2.13. Use the same notation as in Definition 2.11. Suppose that in the definition of $\mathcal{B}$ the requirement that for all but finitely many a, $U_a = X_a$ is dropped. Compare the topology generated by this revised $\mathcal{B}$ with the product topology.

Definition 2.14. Here are some simple, but common, topological spaces.

(a) (n–space) Give $\mathbb{R}^n$ the product topology. This is called the usual topology on $\mathbb{R}^n$. It is generated by the usual metric on $\mathbb{R}^n$.
(b) (the n–sphere) Let S^n consist of all unit vectors in $\mathbb{R}^{n+1}$. Give S^n the subspace topology coming from the usual topology on $\mathbb{R}^{n+1}$.
(c) (the n–torus) Let $A = \{1, \dots, n\}$. Define $T^n = \times_A S^1$. Give T^n the product topology arising from the usual topology on S^1.
(d) (the n–ball) Let B^n consist of all vectors in $\mathbb{R}^{n+1}$ of length less than or equal to 1. Give B^n the subspace topology.
(e) (the n–cube) Let $I = [0, 1] \subset \mathbb{R}$ with the subspace topology. Let $A = \{1, \dots, n\}$ and let $I^n = \times_A I$. Give I^n the product topology.

Exercise 2.15. Prove that, for every n, the n–ball is homeomorphic to the n–cube.

Exercise 2.16. Let $N = (1, 0, \dots, 0) \in S^n$. Let $X = S^n \backslash N$. Give X the subspace topology. Prove that X is homeomorphic to $\mathbb{R}^n$. (Hint: use stereographic projection.)

2.3. **Digression: Equivalence Relations and Groups.**

Definition 2.17. Let X be a set and let $\sim$ be a relation. Then $\sim$ is an **equivalence relation** if and only if for all $x, y, z \in X$

(a) (Reflexive) $x \sim x$
(b) (Symmetric) $x \sim y \Rightarrow y \sim x$
(c) (Transitive) $x \sim y$ and $y \sim z$ implies that $x \sim z$.

For $x \in X$, $[x] = \{y \in X : x \sim y\}$. $[x]$ is called the **equivalence class** of X. The notation $X/\!\sim$ denotes the set of equivalence classes of elements of X.

Example 2.18. Define an equivalence relation $\sim$ on $Q = \mathbb{Z} \times (\mathbb{Z} - \{0\})$ by $(a, b) \sim (m, n)$ if and only if $an = bm$. Then $Q/\sim$ has a natural identification with the rational numbers $\mathbb{Q}$.

Exercise 2.19. (Crushing) Let X be a set and let $A \subset X$. Define an relation $\sim$ on X by $x \sim y$ if and only if either $x = y$ or $x, y \in A$. Show that $\sim$ is an equivalence relation.

Exercise 2.20. (Gluing) Let X and Y be sets and let $A \subset X$ and $B \subset Y$. Assume that $A \cap B = \varnothing$. Let $f\colon A \to B$ be a function. Define a relation $\sim_f$ on $X \cup Y$ by

(a) For all $z \in X \cup Y$, $z \sim_f z$
(b) If $x, z \in A$ and $f(x) = f(z)$ then $x \sim_f z$.
(c) For all $x \in A$, $x \sim_f f(x)$ and $f(x) \sim_f x$.

Prove that $\sim_f$ is an equivalence relation. The set $(X \cup Y)/\sim$ is often denoted by $X \cup_f Y$.

The following few definitions and exercises are a detour from topology into group theory. We will make use of these facts later in the course.

Definition 2.21. A **group** $(G, \cdot)$ is a set G together with a binary operation $\cdot$ such that the following hold:

(a) (Closure) For all $g, h \in G$, $g \cdot h \in G$
(b) (Associative) For all $g, h, k \in G$, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$
(c) (Identity) There exists $e \in G$ such that for all $g \in G$, $g \cdot e = e \cdot g = g$.
(d) (Inverses) For each $g \in G$ there exists $h \in G$ such that $g \cdot h = h \cdot g = e$. The element h is usually denoted g^{-1}.

A group is **abelian** if in addition

(e.) (Commutative) For all $g, h \in G$, $g \cdot h = h \cdot g$.

If $H \subset G$, then H is a **subgroup** of G if H is a group *with the same operation* $\cdot$. When talking abstractly about an arbitrary group $(G, \cdot)$ it is often convenient to write $g \cdot h$ as gh. We will often do this.

Exercise 2.22. Prove that the following are groups. Which are abelian?

(a) $(\mathbb{R}, +)$
(b) $(\mathbb{R}\backslash\{0\}, \cdot)$
(c) $(C_n, \cdot)$ where C_n consists of all complex numbers which are nth roots of 1. That is, numbers of the form

$$e^{2k\pi i/n} \qquad \text{for natural numbers } k \text{ such that } 0 \leq k \leq n-1.$$

This group corresponds to the group of rotations of a regular n–gon in the complex plane.
(d) 2×2 matrices with real entries and non-zero determinant. The operation is matrix multiplication. This group is denoted $GL_2(\mathbb{R})$.
(e) $\mathrm{Homeo}(X)$ for a topological space X. This is the set of homeomorphisms $\{h\colon X \to X\}$ with group operation the composition of functions.

Exercise 2.23. (Cosets) Let $(G, \cdot)$ be a group and let H be a subgroup of G. Define a relation $\sim_H$ on G by $g_1 \sim g_2$ if and only if there exist $h_1, h_2 \in H$ such that $g_1 h_1 = g_2 h_2$.

(a) Prove that $\sim_H$ is an equivalence relation.

An equivalence class $[g]$ is usually denoted gH. It is called a **coset** of H in G. The set $G/\sim_H$ is usually denoted G/H.

(b.) Consider the group $(\mathbb{Z}, +)$ and let $n \in \mathbb{N}$. Prove that

$$n\mathbb{Z} = \{z \in \mathbb{Z} : \text{ there exists } m \in \mathbb{Z} \text{ with } z = nm\}$$

is a subgroup of $\mathbb{Z}$.
(c.) For $n = 5$, list all the elements of $\mathbb{Z}/n\mathbb{Z}$.

Definition 2.24. If G is a group and if H is a subgroup of G then we say that H is **normal** if for all $g \in G$ and for all $h \in H$,

$$g^{-1}hg \in H.$$

Example 2.25. Let H be a normal subgroup of G. Define a binary operation $\circ$ on G/H by

$$g_1 H \circ g_2 H = (g_1 g_2)H.$$

Prove that $\circ$ is well defined and that $(G/H, \circ)$ is a group.

Proof. To prove that $\circ$ is well defined, suppose that $k_1 \in g_1H$ and $k_2 \in g_2H$. We must show that

$$k_1H \circ k_2H = g_1H \circ g_2H.$$

Since equivalence classes partition a set, it is enough to show that $(k_1k_2) \in (g_1g_2)H$. Since $k_1 \in g_1H$, there exists $h_1 \in H$ so that $k_1 = g_1h_1$. Similarly, there exists $h_2 \in H$ so that $k_2 = g_2h_2$. Since H is normal in G, there exists $h \in H$ so that $g_2^{-1}h_1g_2 = h$. Thus,

$$k_1k_2 = g_1h_1g_2h_2 = g_1g_2(g_2^{-1}h_1g_2)h_2 = g_1g_2(hh_2).$$

Since H is a subgroup $hh_2 \in H$. Thus, $k_1k_2 \in (g_1g_2)H$. Consequently, $\circ$ is well defined.

The associativity of $\circ$ follows immediately from the associativity of $\cdot$ (the group operation on G). The identity element of G/H is the coset H. The inverse of gH is $g^{-1}H$. Hence G/H is a group. $\square$

Definition 2.26. Let G and K be groups and let $f\colon G \to K$ be a function. Then f is a **homomorphism** if for all $g_1, g_2 \in G$,

$$f(g_1g_2) = f(g_1)f(g_2).$$

If f is bijective, f is said to be an **isomorphism**.

Exercise 2.27. Let $n \in \mathbb{N}$. Show that the group C_n is isomorphic to the group $\mathbb{Z}/n\mathbb{Z}$. We will often denote any group isomorphic to these groups as $\mathbb{Z}_n$.

Definition 2.28. Let G be a group and let X be a set. An **action** of G upon X is a function $*\colon G \times X \to X$ such that

(a) If e is the identity element of G, then for all $x \in X$, $e * x = x$.
(b) For all $g, h \in G$, $g * (h * x) = (gh) * x$.

If X is a topological space, we will often require, for each g, the map $x \to g * x$ to be continuous. If G is a topological group and if X is a topological space, we will require $*$ to be a continuous function.

Exercise 2.29. For the following groups G, sets X, and functions $*$, show that $*$ is an action of G on X.

(a) Let $G = \mathbb{Z}$, $X = \mathbb{R}$, and $g * x = g + x$.
(b) $G = GL_2(\mathbb{R})$ and $X = \mathbb{R}^2$. The function $*$ is matrix-vector multiplication.
(c) $G = C_n$ and X is a regular n–gon in $\mathbb{C}$ centered at the origin and with one vertex at 1. (The action should rotate X by $2\pi/n$ radians.)

Exercise 2.30. (Quotienting by a group action) Suppose that G is a group and that X is a set such that $*$ is an action of G on X. Define a relation $\sim_*$ on X by $x \sim_* y$ if and only if there exists $g \in G$ such that $g * x = y$. Show that $\sim_*$ is an equivalence relation on X. The set of equivalence classes is often denoted by $G\backslash X$.

Exercise 2.31. For each of the groups G and sets X in Exercise 2.29 describe (geometrically or otherwise) the sets $G\backslash X$.

2.4. **The quotient topology.**

Definition 2.32. Let $(X, \mathcal{X})$ be a topological space and let $\sim$ be an equivalence relation on X. Let $f\colon X \to X/\!\sim$ be the function $f(x) = [x]$. Define a set $U \subset X/\!\sim$ to be open if and only if $f^{-1}(U) \in \mathcal{X}$. The set of all open sets in $X/\!\sim$ is the **quotient topology** on $X/\!\sim$. The function f is called the **quotient function** or **quotient map**.

Exercise 2.33. Show that the quotient topology is, in fact, a topology.

Exercise 2.34. Show that the quotient topology is the largest topology on $X/\!\sim$ for which the quotient function f is continuous.

Definition 2.35. Let X and Y be topological spaces and let $f\colon X \to Y$ be a function. Then f is **open** if the image of every open set is open.

Exercise 2.36. Prove that the quotient function is open.

Exercise 2.37. Suppose that X, Y, Z are topological spaces and that $f\colon X \to Y$, $g\colon X \to Z$, and $h\colon Y \to Z$ are functions such that $g = h \circ f$. Suppose that f is an open function and that g is continuous. Prove that h is continuous.

Exercise 2.38. Consider the set $I = [0, 1]$ with the usual topology. Let $A = \{0, 1\}$ and give I/A the quotient topology. Prove that I/A is homeomorphic to S^1.

Notice that you must prove that I/A with the quotient topology is homeomorphic to the set of unit vectors in $\mathbb{R}^2$ with the subspace topology. **Hint:** Consider

$$g\colon I \to S^1 \qquad \text{defined by} \qquad g(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Show that g gives rise to a well-defined bijection $h\colon I/A \to S^1$. Use Exercise 2.37 to prove that h is continuous. Finally, you need to show that h^{-1} is continuous.

Exercise 2.39. Give $I^2 = [0, 1] \times [0, 1]$ the usual topology. Define an equivalence relation on I^2 by

- $(x, y) \sim (x, y)$ for all $(x, y) \in I^2$
- $(x, 0) \sim (x, 1)$ for all $x \in I$
- $(0, y) \sim (1, y)$ for all $y \in I$.

Prove that $I^2/\sim$ with the quotient topology is homeomorphic to T^2.

Exercise 2.40. Use the quotient topology to give more detail or more certainty to your descriptions in Exercise 2.31.

Exercise 2.41. Let $X = \mathbb{R}^n \backslash \{0\}$. Say that $x_1 \sim x_2$ if there exists $\lambda > 0$ such that $x_1 = \lambda x_2$. Prove that $\sim$ is an equivalence relation and that $X/\sim$ with the quotient topology is homeomorphic to S^{n-1}.

Exercise 2.42. Let $X = \mathbb{R}^n \backslash \{0\}$. Say that $x_1 \sim x_2$ if there exists a real number $\lambda \neq 0$ such that $x_1 = \lambda x_2$. It is a fact that $\sim$ is an equivalence relation. Let $\mathbb{P}^{n-1} = X/\sim$ with the quotient topology. Prove that there is a surjective function $f\colon S^{n-1} \to \mathbb{P}^{n-1}$ such that f is a local homeomorphism. That is, for each point $x \in S^{n-1}$ there is an open set $U \subset S^{n-1}$ such that $f|_U$ is a homeomorphism onto $f(U)$.

Exercise 2.43. Show that S^1 and $\mathbb{P}^1$ are homeomorphic.

It is a fact that for $n > 1$, S^n and $\mathbb{P}^n$ are not homeomorphic.

Exercise 2.44. Define an action of the group $\mathbb{Z}^2$ on the topological space $\mathbb{R}^2$ by

$$(n, m) * (x, y) = (x + n, y + m).$$

Show that $\mathbb{Z}^2 \backslash \mathbb{R}^2$ is homeomorphic to T^2.

Definition 2.45. Let X be a topological space. Let $X_t = X \times \{t\} \subset X \times I$. Give $X \times I$ the quotient topology. The **cone** of X, denoted SX, is the space $(X \times I)/X_1$. The **suspension** of X, denoted SX is $X/\sim$ where for all $(x, s), (y, t) \in X \times I$, $(x, s) \sim (x, s)$ and $(x, s) \sim (y, t)$ if and only if $s = t = 0$ or $s = t = 1$.

Exercise 2.46. Let X be the n–sphere, S^n. Show that SX is homeomorphic to S^{n+1}.

Exercise 2.47. Let $X = B^2 \times \{0\}$ and $Y = B^2 \times \{1\}$. Notice that X and Y are both homeomorphic to B^2. Let $A = S^1 \times \{0\} \subset X$ and $B = S^1 \times \{1\} \subset Y$. Define $f\colon A \to B$ by $f(s, 0) = (s, 1)$. Prove that $Y \cup_f X$ with the quotient topology is homeomorphic to S^2. Generalize, if you can, this result to higher dimensions.

3. Topological Properties

3.1. **Separating points.**

Definition 3.1. A topological space $(X, \mathcal{X})$ is **Hausdorff** (or T_2) if for every $x, y \in X$ with $x \neq y$ there exist disjoint open sets U_x and U_y containing x and y respectively.

Exercise 3.2. Prove that every metric space is Hausdorff.

Exercise 3.3. (Line with two origins) Let $X = (\mathbb{R}\backslash\{0\}) \cup \{a, b\}$. Define a base for a topology on X as follows. Declare the following sets to be open intervals:

(a) (x, y) such that $x < 0$ and $y \leq 0$
(b) (x, y) such that $x \geq 0$ and $y > 0$
(c) $(x, 0) \cup \{a\} \cup (0, y)$ for $x < 0$ and $y > 0$.
(d) $(x, 0) \cup \{b\} \cup (0, y)$ for $x < 0$ and $y > 0$.

Let $\mathcal{X}$ be the topology on X generated by these open intervals.

(a) Show that $(X, \mathcal{X})$ is not Hausdorff.
(b) Show that $X/\{a, b\}$ is homeomorphic to $\mathbb{R}$ and is, therefore, Hausdorff.

Exercise 3.4. Define an equivalence relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

Prove that $\mathbb{R}/\sim$ is non-Hausdorff.

3.2. **Connectedness.**

Definition 3.5. Let $(X, \mathcal{X})$ be a topological space. X is **connected** if the only sets which are both open and closed are X and $\varnothing$. Suppose that $A \subset X$ is a non-empty set which is both open and closed. If, in the subspace topology, A is connected then A is a **connected component** of X.

Exercise 3.6. Prove that $(X, \mathcal{X})$ is not connected if and only if there exist nonempty disjoint open sets $A, B \subset X$ such that $X = A \cup B$. Notice that A and B are both open and closed.

The next exercise is a very useful tool for proving that certain spaces are or are not connected.

Exercise 3.7. Let N be a finite set with the discrete topology. Prove that the number of connected components of a topological space X is at least $|N|$ if and only if there exists a continuous surjection of X onto N.

Exercise 3.8. (a) Prove that the Intermediate Value Theorem is equivalent to the statement that every closed, bounded interval $[a, b] \subset \mathbb{R}$ is connected
(b) Prove that if X and Y are connected topological spaces then $X \times Y$ is connected.
(c) Prove that if X and Y are homeomorphic topological spaces and if X is connected then so is Y.

The previous exercise indicates that it might be useful to know that the Intermediate Value Theorem is true. A proof is given in Section 5. From now on, you may assume that the Intermediate Value Theorem is true.

Exercise 3.9. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Prove that if $(X, \mathcal{X})$ is connected and if there exists a surjective continuous function from X to Y then Y is connected. Conclude that if X is connected and if $\sim$ is an equivalence relation on X then $X/\!\sim$ is connected.

Exercise 3.10. Prove that S^1 and T^n for $n \geq 2$ are connected.

Definition 3.11. Let $(X, \mathcal{X})$ be a topological space. A **path** in X is a continuous function $p\colon\ I \to X$. We say that p is a path from $p(0)$ to $p(1)$. A space $(X, \mathcal{X})$ is **path-connected** if for all $x, y \in X$ there exists a path from x to y.

Exercise 3.12. Prove that if X is path-connected then it is connected.

Exercise 3.13. (a) Prove that $\mathbb{R}^n$ is connected for $n \geq 1$.
(b) Let $y \in \mathbb{R}^n$ for some $n \geq 2$. Prove that $\mathbb{R}^n - \{y\}$ is connected (in the subspace topology).
(c) Let $x \in \mathbb{R}$. Prove that $\mathbb{R} - \{x\}$ is not connected (in the subspace topology).

Exercise 3.14. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Prove the following.

(a) Suppose that $f\colon\ X \to Y$ is a homeomorphism. Let $x \in X$. Give $X - \{x\}$ and $Y - \{f(x)\}$ the subspace topologies. Prove that $X - \{x\}$ is homeomorphic to $Y - \{f(x)\}$.
(b) Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.
(c) Prove that I is not homeomorphic to $(0, 1)$.
(d) Prove that I is not homeomorphic to S^1.
(e) Prove that the comb space (Exercise 1.15) is not homeomorphic to the upper half space with the usual topology.

Definition 3.15. $(X, \mathcal{X})$ is **locally connected** if for each point $x \in X$ and each open set $U \subset X$ such that $x \in U$, there exists an open set $V \subset U$ such that $x \in V$ and V is connected (with the subspace topology).

Exercise 3.16. (The topologists' sine curve) Let S be the graph of $y = \sin(1/x)$ for $x > 0$ in $\mathbb{R}^2$. Let

$$J = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$$

Define $\Gamma = S \cup J$. Γ (with the subspace topology) is the topologists' sine curve.

(a) Show that Γ is connected.
(b) Show that Γ is not locally connected.
(c) Show that Γ is not path-connected.
(d) Modify the construction of Γ to produce an example of a space which is path connected but not locally connected.

3.3. **Covers and Compactness.**

Definition 3.17. Let $(X, \mathcal{X})$ be a topological space. A **cover** of X is a collection of sets $\{U_\alpha : \alpha \in A\}$ (for some indexing set A) such that $\cup U_\alpha = X$. If all the sets U_α are open, it is an **open cover**. If they are all closed, it is a **closed cover**. A **subcover** is a subset of $\{U_\alpha : \alpha \in A\}$ which is also a cover of X.

Warning: In algebraic topology the term "cover" has a completely different meaning.

Definition 3.18. A topological space $(X, \mathcal{X})$ is **second countable** if and only if there is a countable base for the topology.

Exercise 3.19. Show that a metric space (X, d) is second countable if every open cover has a countable subcover.

Definition 3.20. Let $(X, \mathcal{X})$ be a topological space. X is **compact** if every open cover of X has a finite subcover.

Exercise 3.21. Prove that if a topological space has a finite number of points then it is compact.

Exercise 3.22. Prove that $\mathbb{R}^n$ is not compact for $n \geq 1$.

Exercise 3.23. Let $(X, \mathcal{X})$ be a Hausdorff topological space and suppose that $A \subset X$ is a compact subspace. Prove that A is closed.

Exercise 3.24. Let X and Y be topological spaces. Suppose that X is compact and $f\colon X \to Y$ is continuous. Show that $f(X)$ is compact (with the subspace topology in Y).

Exercise 3.25. Prove that if $A \subset X$ is closed and if X is compact, then A is compact.

Exercise 3.26. Prove that if X is compact and if Y is Hausdorff and if $f\colon X \to Y$ is a continuous bijection then f is a homeomorphism.

The next two theorems are extremely important. The first is proven in Section 3.4 and the second in Section 5.

Theorem 3.27. If X and Y are compact topological spaces then $X \times Y$ (with the product topology) is compact.

Theorem 3.28. The interval $[0, 1] \subset \mathbb{R}$ is compact.

Exercise 3.29. Prove that if $X_1, \ldots, X_n$ are compact topological spaces, then $\times_{i=1}^n X_i$ is compact.

Tychonoff's theorem states that the product of an infinite number of compact topological spaces is also compact. Tychonoff's theorem is much harder to prove than the previous exercise.

Definition 3.30. Suppose that (X, d) is a metric space and that $A \subset X$. Then A is **bounded** if there exists N such that for all $x, y \in A$, $d(x, y) \leq N$.

Exercise 3.31. Suppose that A is a bounded subset of a metric space (X, d). Prove that for all $x_0 \in X$, there exists $r > 0$ such that $A \subset B_r(x_0)$.

Exercise 3.32. (Heine-Borel Theorem) Suppose that $X \subset \mathbb{R}^n$ is a closed and bounded subset. Then X is compact (with the subspace topology).

Exercise 3.33. (Extreme Value Theorem) Let X be a compact topological space and suppose that $f\colon\ X \to \mathbb{R}$ is continuous. Then there is $x \in X$ such that for all $y \in X$, $f(x) \geq f(y)$. The number $f(x)$ is the **global maximum** of f. Similarly, show that f attains its global minimum.

Exercise 3.34. Prove that S^n is compact. Use this to show that, at a fixed moment in time, someplace on earth has a temperature that is equal to or larger than the temperature at every other place on earth.

Exercise 3.35. Prove that S^n and $\mathbb{R}^m$ are not homeomorphic for any $n, m \geq 0$.

Exercise 3.36. Suppose that X is a Hausdorff space with infinitely many points. Let $x \in X$. Prove that $X \backslash \{x\}$ is not compact.

3.4. **Compactness of the product of two compact topological spaces.**

Definition 3.37. If Z and Y are compact topological spaces and if $f\colon\ Z \to Y$ is a function then f is **proper**, if for each compact set $C \subset Y$, $f^{-1}(C)$ is compact. The function f is **open**, if for each open set $U \subset Z$, the set $f(U)$ is open in Y.

Lemma 3.38. Suppose that Z and Y are topological spaces and that $f\colon\ Z \to Y$ is an open function. Suppose that for each $y \in Y$, $f^{-1}(y)$ is compact in X. Then f is a proper function.

Proof. Let $C \subset Y$ be compact. We wish to show that $f^{-1}(C)$ is compact in Z. Let $\{U_\alpha : \alpha \in A\}$ be an open cover, with A an index set. For each $y \in C$, $f^{-1}(y)$ is compact. Let $A_y \subset A$ be a finite subset so that $\{U_\alpha : \alpha \in A_y\}$ is a finite cover of $f^{-1}(y)$. Since each U_α is open, the set

$$W_y = \bigcup_{\alpha \in A_y} U_\alpha$$

is open in Z.

Since f is an open function, the image of a closed set is closed. Hence, $f(Z - W_y)$ is a closed set in Y. Thus,

$$V_y = Y - f(Z - W_y)$$

is an open set in Y.

Claim: For each $y \in C$, $y \in V_y$. Suppose that y is not in V_y. Then $y \in f(Z - W_y)$. This means that there exists $z \in f^{-1}(y)$ such that $z \in Z - W_y$. However, there exists $\alpha \in A_y$ so that $z \in U_\alpha$. Furthermore, this U_α is a subset of W_y. So $z \in U_\alpha \subset W_y$. This means that z is not in $Z - W_y$, a contradiction.

Hence, $\{V_y\}$ is an open cover of the compact set C. Thus, there exist points $y_1, \ldots, y_n \in C$ so that $\{V_{y_i} : 1 \leq i \leq n\}$ is a finite open cover of C.

Claim: For each y_i, $f^{-1}(V_{y_i}) \subset W_{y_i}$. Suppose that $z \in f^{-1}(V_{y_i})$. Then, $f(z) \in Y - f(Z - W_y)$. Hence z is not in $Z - W_y$ which means that $z \in W_y$. □

Since $C \subset \bigcup V_{y_i}$, we have

$$f^{-1}(C) \subset \bigcup_{i=1}^{n} f^{-1}(V_{y_i}) \subset \bigcup_{i=1}^{n} W_{y_i} = \bigcup_{i=1}^{n} \bigcup_{\alpha \in A_{y_i}} U_\alpha.$$

Thus,

$$\{U_\alpha : \text{ there exists } 1 \leq i \leq n \text{ so that} \alpha \in A_{y_i}\}.$$

This is a finite subset of $\{U_\alpha : \alpha \in A\}$ and it covers $f^{-1}(C)$, so $f^{-1}(C)$ is compact and f is proper. □

Lemma 3.39. Let X and Y be topological spaces. Then the projection $\pi_Y : X \times Y \to Y$ is an open function.

Proof. Let $W \subset X \times Y$ be an open set. Since products of open sets from X and Y form a basis of the product topology, there exist open sets $U_\alpha \subset X$ and $V_\alpha \subset Y$ for α in some index set A so that $W = \bigcup U_\alpha \times V_\alpha$. Then,

$$\pi_Y(W) = \pi_Y(\bigcup U_\alpha \times V_\alpha) = \bigcup \pi_Y(U_\alpha \times V_\alpha) = \bigcup V_\alpha$$

This last set is open in Y. □

Lemma 3.40. Suppose that X and Y are topological spaces and that X is compact. Then for each $y \in Y$, $\pi_Y^{-1}(y) \subset X \times Y$ is compact.

Proof. For each $y \in Y$,

$$\pi_Y^{-1}(y) = X \times \{y\}$$

This is homeomorphic to X and so is compact. □

Theorem 3.27. Suppose that X and Y are compact topological spaces. Then $X \times Y$ is compact.

Proof. The projection $\pi_Y \colon X \times Y \to Y$ is open and the inverse image of a point in Y is compact. Thus, by Lemma 3.38, π_Y is proper. Since Y is compact, $\pi_Y^{-1}(Y) = X \times Y$ is compact. □

4. SEQUENCES

Definition 4.1. A **sequence** in a set X is a function $f \colon N \to X$ where N is an infinite subset of $\mathbb{N}$. We often write x_i instead of $f(i)$ and $(x_i : i \in N)$ for f. If $N = \mathbb{N}$, we will write $(x_i)_i$.

Warning: Confusing the sequence $(x_i)_i$ which is a function with the set $\{x_i\}_i \subset X$ can lead to trouble. However, the difference is often obscured in practice.

Definition 4.2. Suppose that $(x_i : i \in N)$ is a sequence in a set X. If N' is an infinite subset of N, then the sequence $(x_i : i \in N')$ is a **subsequence** of $(x_i : i \in N)$.

Definition 4.3. Let $\mathbf{x} = (x_i : i \in N)$ be a sequence in a topological space X. A point $l \in X$ is a **limit** of $\mathbf{x}$ if for every open set U containing l there is an $M \in \mathbb{N}$ such that for every $n \geq M$, $x_n \in U$. We say that $\mathbf{x}$ **converges** to l. A sequence $\mathbf{x}$ is **convergent** if there exists a limit for $\mathbf{x}$.

Exercise 4.4.

(a) Prove that if X has the indiscrete topology then every point of X is a limit of every sequence in X.

(b) Prove that if X has the discrete topology then the only sequences that have limits are sequences which are either finite or eventually constant.

(c) Prove that if a sequence in a Hausdorff space has a limit point then that limit point is unique.

Definition 4.5. Let X be a topological space and let $A \subset X$. A **limit point** of A is a point $x \in X$ such that for every open set containing x, $U \cap A$ contains a point other than x.

Exercise 4.6. Suppose that $(x_i : i \in N)$ is a sequence in a metric space (X, d). Determine necessary and sufficient conditions for a point l to be both a limit of the sequence $(x_i : i \in N)$ and a limit point of the set $\{x_i : i \in N\}$.

Exercise 4.7. Let X be a metric space and $A \subset X$.

(a) Suppose that l is a limit point of A. Prove that there exists a sequence $\mathbf{x} = \{x_i : i \in N\}$ in A such that l is a limit of $\mathbf{x}$.
(b) Suppose that $\mathbf{x}$ is a sequence in A which converges to a point $l \in X$. Prove that either l is a limit point of A or that the sequence $\mathbf{x}$ is eventually constant. (That is, there exists $M \in N$ such that for all $n \geq M$, $x_m = l$.)

Definition 4.8. Let X be a metric space and suppose $A \subset X$. Let $\overline{A}$ be the union of A and the set of its limit points. $\overline{A}$ is called the **closure** of A in X. Notice that it depends not only on A but also on X.

Exercise 4.9. Let X be a metric space and let $A \subset X$.

(a) Prove that $\overline{A}$ is a closed set.
(b) Prove that $\overline{A}$ is the smallest closed set containing A. That is, if $A \subset V \subset X$ and if V is closed, then $\overline{A} \subset V$.
(c) Prove that if $A \subset X$ is closed and if l is a limit point of A then $l \in A$. (That is, then $A = \overline{A}$.)
(d) (Challenging!) Do this exercise again, but with the hypothesis that X is a metric space replaced with the hypothesis that X is a Hausdorff space.

Definition 4.10. A function $f\colon\ X \to Y$ is said to be **sequentially continuous** if whenever a sequence $(x_i : i \in N)$ in X converges to $L \in X$, the sequence $(f(x_i) : i \in N)$ in Y converges to $f(L) \in Y$.

Exercise 4.11. Let X and Y be metric spaces and let $f\colon\ X \to Y$ be a function. Prove that f is continuous if and only if it is sequentially continuous.

Definition 4.12. A space $(X, \mathcal{X})$ is **sequentially compact** if every sequence in X contains a convergent subsequence.

Exercise 4.13. Prove that if a metric space is compact then it is sequentially compact.

Definition 4.14. Let (X, d) be a metric space and let $\{x_i\} \subset X$ be a sequence. Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) < \epsilon$. Then the sequence $\{x_i\}$ is a **Cauchy sequence**. If every Cauchy sequence in X converges then (X, d) is a **complete** metric space.

Exercise 4.15. (a) Prove that if (X, d) is a compact metric space then (X, d) is complete.
(b) Give an example of a metric space which is complete but noncompact.

Definition 4.16. If $A \subset X$ is a subset of a metric space (X, d), then the **diameter** of A is

$$\operatorname{diam}(A) = \sup\{d(p, q) : p, q \in A\}$$

Exercise 4.17. Prove that a compact subset of a metric space has finite diameter.

Exercise 4.18. Give an example of two homeomorphic metric spaces such that one of them has finite diameter and the other infinite diameter.

Exercise 4.19. (Lebesgue Covering Lemma[1]) Let (X, d) be a compact metric space and let $\{U_\alpha\}$ be an open covering of X. Then there exists $\delta > 0$ such that for each set $A \subset X$ with $\operatorname{diam}(A) < \delta$ there exists α such that $A \subset U_\alpha$. The number δ is the **Lebesgue number** for $\{U_\alpha\}$.

Exercise 4.20. (Contraction maps[2]) Let (X, d) be a complete metric space. Let $f\colon X \to X$. Suppose that there exists $0 \leq \lambda < 1$ such that for every $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Prove that there is a unique $x \in X$ such that $f(x) = x$. The point x is called a **fixed point** of f.

5. Connectedness and Compactness of Intervals

Definition 5.1. Suppose that $A \subset \mathbb{R}$. An **upper bound** for A is a number β such that for all $a \in A$, $a \leq \beta$. The **least upper bound** or **supremum** of A is an upper bound β for A such that if β' is any other upper bound for A then $\beta \leq \beta'$. We write $\sup A = \beta$ if β exists. If A has no upper bound we write $\sup A = \infty$.

A **lower bound** for A is a number α such that for all $a \in A$, $\alpha \leq a$. The **greatest lower bound** or **infimum** of A is a lower bound α for A such that if α' is an other lower bound for A then $\alpha' \leq \alpha$. We write $\inf A = \alpha$ if α exists. If A has no lower bound we write $\inf A = -\infty$.

The theory of real numbers, usually covered in an analysis class, guarantees that if A is non-empty and has an upper bound then $\sup A$ exists and if A is non-empty and has a lower bound then $\inf A$ exists.

Lemma 5.2. Suppose that $A \subset \mathbb{R}$ is a non-empty bounded set. If A is closed then $\sup A \in A$ and $\inf A \in A$.

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[1] The statement comes from Bredon's Topology and Geometry

[2] Taken from Browder's Mathematical Analysis

Theorem 3.28. The interval $[0,1]$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of $[0,1]$. Define

$$S = \{s \in [0,1] : \text{ there is a finite subset of } \mathcal{U} \text{ which covers } [0,s]\}.$$

Let b be the least upper bound for S.

Claim : $S = [0,b)$ or $S = [0,b]$
To see this, suppose that $s \in S$. Let $\mathcal{U}'$ be a finite subset of $\mathcal{U}$ which covers $[0,s]$. Then for all $s' < s$, $\mathcal{U}'$ also covers $[0,s']$. Hence, if $s \in S$, then $[0,s] \subset S$. □

Claim: $S = [0,b]$.
Suppose not; that is, suppose that $S = [0,b)$. Since $\mathcal{U}$ is an open cover of $[0,1]$, there exists an open set $U \in \mathcal{U}$ so that $b \in \mathcal{U}$. Since U is an open set in $[0,1]$, there exists $\epsilon > 0$ so that the interval $(b-\epsilon, b]$ is a subset of U. Since b is the least upper bound for S, the number $b-\epsilon/2$ is contained in S. Let $\mathcal{U}'$ be a finite subset of $\mathcal{U}$ which covers $[0,b-\epsilon/2]$. Then $\mathcal{U}' \cup \{U\}$ is a finite subset of U' which covers $[0,b]$. Hence, $b \in S$ and so, $S = [0,b]$. □

Claim: $S = [0,1]$. Suppose not. That is, suppose that $b < 1$. Let $\mathcal{U}'$ be a finite subset of $\mathcal{U}$ which covers $S = [0,b]$. Since each set of $\mathcal{U}$ is open in $[0,1]$, there exists a set $U \in \mathcal{U}$ so that $b \in U$. Since U is open and since $b < 1$, there exists $\epsilon > 0$ so that $(b, b+\epsilon) \subset U$. Thus, $\mathcal{U}'$ is a finite subset of $\mathcal{U}$ which covers $[0,b+\epsilon/2]$. This implies that $b+\epsilon/2 \in S$. But, b is the least upperbound for S and so this is impossible. □

Thus, $[0,1]$ is compact. □

Intermediate Value Theorem. Let $[a,b] \subset \mathbb{R}$ and suppose that $f\colon [a,b] \to \mathbb{R}$ is a continuous function such that $f(a) \neq f(b)$. If y is between $f(a)$ and $f(b)$ then there exists $\beta \in [a,b]$ so that $f(\beta) = y$.

Proof. To begin, assume that $f(a) < y < f(b)$. Let $A = f^{-1}([f(a), y])$. Since $a \in A$, A is a bounded, non-empty set. Hence, $\beta = \sup A$ exists. Since $[f(a), y]$ is closed, and f is continuous, A is closed. By Lemma 5.2, $\beta \in A$. This implies that $f(\beta) \leq y$. Since $b \notin A$ and since $\beta = \sup A$, the interval $J = (\beta, b)$ is a non-empty open interval disjoint from A. Then the sequence, $x_n = \beta + 1/n$ is a sequence of points not in A converging to β. Since $x_n \notin A$, $f(x_n) > y$ for all n. Since f is continuous, $f(x_n)$ converges to $f(\beta)$. Thus, $f(x_n) > y$ for all n implies that $f(\beta) \geq y$. Consequently, $f(\beta) = y$.

The case when $f(b) < y < f(a)$ is similar and we omit the proof. □