Point-Set Topology

1. Topological Spaces and Continuous Functions

Definition 1.1. Let X be a set and $\mathcal{T}$ a subset of the power set $\mathcal{P}(X)$ of X. Then $\mathcal{T}$ is a **topology** on X if and only if all of the following hold

(a) $\varnothing \in \mathcal{T}$
(b) $X \in \mathcal{T}$
(c) (Arbitrary unions) If $A_\alpha \in \mathcal{T}$ for α in some index set I then
$$\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}.$$
(d) (Finite intersections) If $B_\beta \in \mathcal{T}$ for β in some finite set I then
$$\bigcap_{\beta \in I} B_\beta \in \mathcal{T}.$$

If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is a **topological space**. Sometimes, if $\mathcal{T}$ is understood we say that X is a topological space. If $U \in \mathcal{T}$ then U is an **open** set. If $V \subset X$ and $X \backslash V$ is open, then V is **closed**. Notice that in any topological space $(X, \mathcal{T})$, both X and $\varnothing$ are sets which are both open and closed. It is possible for a set to be neither open nor closed.

Exercise 1.2. Show that the Finite intersections axiom is equivalent to:

- If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.

Exercise 1.3. Let X be any set. Prove that the following are topologies on X

(a) (the discrete topology) $\mathcal{T} = \mathcal{P}(X)$
(b) (the indiscrete topology) $\mathcal{T} = \{\varnothing, X\}$

Exercise 1.4. Let $X = \{a, b, c, d\}$. Find all topologies on X containing five or fewer sets and prove that your list is complete.

Exercise 1.5. Let $(X, \mathcal{X})$ be a topological space. Prove that a set $U \subset X$ is open if and only if for all $x \in U$ there exists an open set U_x such that $x \in U_x$ and $U_x \subset U$.

Definition 1.6. Let $(X, \mathcal{T})$ be a topological space. A set $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ if each element of $\mathcal{T}$ can be written as the union of elements of $\mathcal{B}$.

Definition 1.7. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. Let $\mathcal{T}$ be the smallest topology on X containing $\mathcal{B}$. That is, $\mathcal{B} \subset \mathcal{T}$ and if $\mathcal{T}'$ is a topology on X such that $\mathcal{B} \subset \mathcal{T}'$ then $\mathcal{T} \subset \mathcal{T}'$. We say that $\mathcal{B}$ **generates** the topology $\mathcal{T}$.

Exercise 1.8. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. Prove that the topology generated by $\mathcal{B}$ exists.

Exercise 1.9. Let $X = \mathbb{R}$ and let $\mathcal{B}$ consist of all intervals in $\mathcal{R}$ of the form (a, b) with $a < b$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. $\mathcal{T}$ is called the **usual** topology on $\mathbb{R}$. You do not need to give completely rigourous answers to the following questions.

(a) Describe the sets in $\mathcal{T}$.
(b) Give an example of a set in $\mathbb{R}$ which is neither open nor closed.

Definition 1.10. Let X be a set and let $d\colon X \times X \to \mathbb{R}$ be a function such that

(a) (Positive) For all $x, y \in X$, $d(x, y) \geq 0$
(b) (Definite) $d(x, y) = 0$ if and only if $x = y$.
(c) (Symmetric) For all x, y, $d(x, y) = d(y, x)$.
(d) (Triangle inequality) For all $x, y, z \in \mathbb{R}$, $d(x, z) \leq d(x, y) + d(y, z)$
The function d is a **metric** on X and (X, d) is a metric space.

Exercise 1.11. Prove that the following are metrics on $\mathbb{R}$.

(a) (the usual metric) $d(x, y) = |x - y|$
(b) (the discrete metric) $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

Exercise 1.12. Prove that the following are metrics on $\mathbb{R}^n$. In each $||x||$ denotes the length of the vector $x \in \mathbb{R}^n$. Also, let x_i denote the ith component of x.

(a) (the usual metric) $d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$
(b) (the sup metric) $d(x, y) = \max_i\{|x_i - y_i|\}$
(c) (the taxicab metric) $d(x, y) = \sum_i |x_i - y_i|$

Exercise 1.13. (The comb metric) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$. Consider the following function $d\colon X \times X \to \mathbb{R}$:

$$d((x_1, x_2), (y_1, y_2)) = \begin{cases} 0 & \text{if } (x_1, x_2) = (y_1, y_2) \\ |x_2 - y_2| & \text{if } x_1 = y_1 \\ |x_2| + |y_2| + |x_1 - y_1| & \text{if } x_1 \neq y_1 \end{cases}$$

Prove that d is a metric and describe the shortest path between two points in X.

Definition 1.14. Let (X, d) be a metric space. For $x \in X$ and $\epsilon > 0$ let

$$B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}.$$

Let $\mathcal{T}$ be the topology generated by

$$\{B_\epsilon(x) : x \in X \text{ and } \epsilon > 0\}.$$

We say that d **generates** the topology $\mathcal{T}$.

Exercise 1.15. Let $\mathcal{T}$ be the topology generated by d. Show that the metric balls $B_\epsilon(x)$ are a base for $\mathcal{T}$.

Exercise 1.16. Prove that the usual metric on $\mathbb{R}$ generates the usual topology on $\mathbb{R}$.

Exercise 1.17. Prove that the usual metric, the sup metric, and the taxicab metric all generate the same topology of $\mathbb{R}^n$.

Exercise 1.18. The usual metric on $\mathbb{R}^2$ restricts to be a metric on the closed upper half space of $\mathbb{R}^2$ (i.e. $\{(x, y) : y \geq 0\}$). Does the usual metric on the closed upper half space generate the same topology as the comb metric? (See Exercise 1.13.)

Definition 1.19. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Let $f\colon X \to Y$ be a function. If $U \subset Y$, let

$$f^{-1}(U) = \{x \in X : f(x) \in U\}.$$

Define f to be continuous if and only if for every open set $U \subset Y$ the set $f^{-1}(U)$ is open in X.

Exercise 1.20. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces and let $f\colon X \to Y$ be a function.

(a) Suppose that there exists $c \in Y$ such that $f(x) = c$ for all $x \in X$. (That is, f is a constant function.) Prove that f is continuous.
(b) Suppose that $\mathcal{X}$ is the discrete topology. Prove that f is continuous.
(c) Suppose that $\mathcal{Y}$ is the indiscrete topology. Prove that f is continuous.

Exercise 1.21. In analysis courses a function $f\colon \mathbb{R} \to \mathbb{R}$ is defined to be continuous at $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. The function is continuous if it is continuous at every point $x_0 \in \mathbb{R}$.

Prove that if $\mathbb{R}$ has the usual topology, f is continuous in the analysis sense if and only if it is continuous in the sense of Definition 1.19.

Exercise 1.22. (The composition of two continuous functions is continuous) Let $(X, \mathcal{X})$, $(Y, \mathcal{Y})$, and $(Z, \mathcal{Z})$ be topological spaces. Let $f\colon X \to Y$ be surjective and continuous. Let $g\colon Y \to Z$ be continuous. Prove that $g \circ f$ is continuous.

Definition 1.23. Let X and Y be sets. A function $f\colon\ X \to Y$ is

(a) **injective** if for every $x_1 \neq x_2 \in X$ $f(x_1) \neq f(x_2)$
(b) **surjective** if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
(c) **bijective** if f is injective and surjective

Recall that if $f\colon\ X \to Y$ is bijective there exists an inverse function

$$f^{-1}\colon\ Y \to X.$$

Definition 1.24. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. A function $f\colon\ X \to Y$ is a **homeomorphism** if f is bijective and both f and f^{-1} are continuous. We say that X and Y are **homeomorphic** if there exists a homeomorphism between them.

Exercise 1.25. Let $\mathbb{R}$ have the usual topology. Let $a < b$ and let $I = (a, b) \subset \mathbb{R}$. The usual metric on $\mathbb{R}$ restricts to a metric on I. Give I the topology generated by this metric. Show that $\mathbb{R}$ and I (with these topologies) are homeomorphic.

Exercise 1.26. Find all sets X such that X with the discrete topology is homeomorphic to X with the indiscrete topology.

2. Constructing Topological Spaces

2.1. **The subspace topology.**

Definition 2.1. Let $(X, \mathcal{X})$ be a topological space and let $Y \subset X$. Define

$$\mathcal{Y} = \{U \cap Y : U \in \mathcal{X}\}.$$

$\mathcal{Y}$ is called the **subspace topology** on Y.

Exercise 2.2. Prove that the subspace topology is a topology.

Exercise 2.3. Let (X, d) be a metric space and let $Y \subset X$. Notice that $d|_Y$ is a metric on Y. Show that $d|_Y$ generates the subspace topology on Y.

2.2. **The product topology.**

Definition 2.4. Let X and Y be sets. The **product** $X \times Y$ is defined as the set of all ordered pairs (x, y) such that $x \in X$ and $Y \in Y$. For sets $X_1, X_2, \ldots, X_n$ the product $\times X_i$ is defined to be the set of n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$. For a countable collection of sets X_i for $i \in \mathbb{N}$, the product $\times X_i$ is defined to be the set of sequences $(x_1, x_2, \ldots)$ with $x_i \in X_i$.

Definition 2.5. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Let $Z = X \times Y$. Let

$$\mathcal{B} = \{U \times V : U \in \mathcal{X} \text{ and } V \in \mathcal{Y}\}.$$

The **product topology** on $\mathbb{Z}$ is the topology generated by $\mathcal{B}$.

Exercise 2.6. Explain why $\mathcal{B}$ in the definition above is usually not itself a topology on Z.

Exercise 2.7. Let $(X, \mathcal{X})$, $(Y, \mathcal{Y})$ be topological spaces. Give $X \times Y$ and $Y \times X$ the product topologies. Prove that $X \times Y$ is homeomorphic to $Y \times X$.

Exercise 2.8. Show that the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the topology generated by the usual metric. (Hint: Use Exercise 1.17.)

Exercise 2.9. Let X and Y be topological spaces and define a function $\mathrm{pr}_X : X \times Y \to X$ by $\mathrm{pr}_X(x, y) = x$. Define pr_Y similarly. Show that the product topology on $X \times Y$ is the smallest topology on $X \times Y$ for which pr_X and pr_Y are continuous. The functions pr_X and pr_Y are called **projections**.

Definition 2.10. Let A be a finite or countably infinite set and let $(X_a, \mathcal{X}_a)$ be a topological space for each $a \in A$. Let $Z = \times_{a \in A} X_a$. Let

$$\mathcal{B} = \{\times_{a \in A} U_a : U_a \in \mathcal{X}_a \text{ and for all but finitely many } a, U_a = X_a\}.$$

The **product topology** on Z is the topology generated by $\mathcal{B}$. In fact, this definition makes sense for sets A which are uncountable, but we will never need that notion.

Exercise 2.11. Use the same notation as in Definition 2.10. Prove that the product topology on Z is the smallest topology for which the projections are continuous. (The definition of projection in this context should be obvious.) If you like, you may simply describe how to adapt your proof from Exercise 2.9.

Exercise 2.12. Use the same notation as in Definition 2.10. Suppose that in the definition of $\mathcal{B}$ the requirement that for all but finitely many a, $U_a = X_a$ is dropped. Compare the topology generated by this revised $\mathcal{B}$ with the product topology.

Definition 2.13. Here are some simple, but common, topological spaces.

(a) (n–space) Give $\mathbb{R}^n$ the product topology. This is called the usual topology on $\mathbb{R}^n$. It is generated by the usual metric on $\mathbb{R}^n$.
(b) (the n–sphere) Let S^n consist of all unit vectors in $\mathbb{R}^{n+1}$. Give S^n the subspace topology coming from the usual topology on $\mathbb{R}^{n+1}$.
(c) (the n–torus) Let $A = \{1, \dots, n\}$. Define $T^n = \times_A S^1$. Give T^n the product topology arising from the usual topology on S^1.
(d) (the n–ball) Let B^n consist of all vectors in $\mathbb{R}^{n+1}$ of length less than or equal to 1. Give B^n the subspace topology.
(e) (the n–cube) Let $I = [0, 1] \subset \mathbb{R}$ with the subspace topology. Let $A = \{1, \dots, n\}$ and let $I^n = \times_A I$. Give I^n the product topology.

Exercise 2.14. Prove that, for every n, the n–ball is homeomorphic to the n–cube.

Exercise 2.15. Let $N = (1, 0, \dots, 0) \in S^n$. Let $X = S^n \backslash N$. Give X the subspace topology. Prove that X is homeomorphic to $\mathbb{R}^n$. (Hint: use stereographic projection.)

2.3. **The quotient topology.**

Definition 2.16. Let X be a set and let $\sim$ be a relation. Then $\sim$ is an **equivalence relation** if and only if for all $x, y, z \in X$

(a) (Reflexive) $x \sim x$
(b) (Symmetric) $x \sim y \Rightarrow y \sim x$
(c) (Transitive) $x \sim y$ and $y \sim z$ implies that $x \sim z$.

For $x \in X$, $[x] = \{y \in X : x \sim y\}$. $[x]$ is called the **equivalence class** of X. The notation $X/\sim$ denotes the set of equivalence classes of elements of X.

Exercise 2.17. (Crushing) Let X be a set and let $A \subset X$. Define an relation $\sim$ on X by $x \sim y$ if and only if either $x = y$ or $x, y \in A$. Show that $\sim$ is an equivalence relation.

Exercise 2.18. (Gluing) Let X and Y be sets and let $A \subset X$ and $B \subset Y$. Assume that $A \cap B = \varnothing$. Let $f\colon A \to B$ be a function. Define a relation $\sim_f$ on $X \cup Y$ by

(a) For all $z \in X \cup Y$, $z \sim_f z$
(b) If $x, z \in A$ and $f(x) = f(z)$ then $x \sim_f z$.
(c) For all $x \in A$, $x \sim_f f(x)$ and $f(x) \sim_f x$.

Prove that $\sim_f$ is an equivalence relation. The set $(X \cup Y)/\sim$ is often denoted by $X \cup_f Y$.

The following few definitions and exercises are a detour from topology into group theory. We will make use of these facts later in the course.

Definition 2.19. A **group** $(G, \cdot)$ is a set G together with a binary operation $\cdot$ such that the following hold:

(a) (Closure) For all $g, h \in G$, $g \cdot h \in G$
(b) (Associative) For all $g, h, k \in G$, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$
(c) (Identity) There exists $e \in G$ such that for all $g \in G$, $g \cdot e = e \cdot g = g$.
(d) (Inverses) For each $g \in G$ there exists $h \in G$ such that $g \cdot h = h \cdot g = e$. The element h is usually denoted g^{-1}.

A group is **abelian** if in addition

(e.) (Commutative) For all $g, h \in G$, $g \cdot h = h \cdot g$.

Exercise 2.20. Prove that the following are groups. Which are abelian?

(a) $(\mathbb{R}, +)$

(b) $(\mathbb{R}\backslash\{0\}, \cdot)$

(c) $(C_n, \cdot)$ where C_n consists of all complex numbers which are nth roots of 1. That is, numbers of the form

$$e^{2k\pi i/n} \qquad \text{for natural numbers } k \text{ such that } 0 \leq k \leq n-1.$$

This group corresponds to the group of rotations of a regular n–gon in the complex plane.

(d) 2×2 matrices with real entries and non-zero determinant. The operation is matrix multiplication. This group is denoted $GL_2(\mathbb{R})$.

(e) $\mathrm{Homeo}(X)$ for a topological space X. This is the set of homeomorphisms $\{h\colon X \to X\}$ with group operation the composition of functions.

Definition 2.21. Let $(G, \cdot)$ be a group and let $H \subset G$. Then H is a **subgroup** of G if $(H, \cdot)$ is a group.

Exercise 2.22. (Quotient Groups) Let $(G, \cdot)$ be a group and let H be a subgroup of G. Define a relation $\sim_H$ on G by $g_1 \sim g_2$ if and only if there exist $h_1, h_2 \in H$ such that $g_1 h_1 = g_2 h_2$.

(a) Prove that $\sim_H$ is an equivalence relation.

An equivalence class $[g]$ is usually denoted gH. The set $G/\sim_H$ is usually denoted G/H.

(b.) Consider the group $(\mathbb{Z}, +)$ and let $n \in \mathbb{N}$. Prove that

$$n\mathbb{Z} = \{z \in \mathbb{Z} : \text{ there exists } m \in \mathbb{Z} \text{ with } z = nm\}$$

is a subgroup of $\mathbb{Z}$.

(c.) For $n = 5$, list all the elements of $\mathbb{Z}/n\mathbb{Z}$.

When talking abstractly about an arbitrary group $(G, \cdot)$ it is often convenient to write $g \cdot h$ as gh. We will often do this.

Definition. If G is a group and if H is a subgroup of G then we say that H is **normal** if for all $g \in G$ and for all $h \in H$,

$$g^{-1}hg \in H.$$

Exercise. Let H be a normal subgroup of G. Define a binary operation $\circ$ on G/H by

$$g_1 H \circ g_2 H = (g_1 g_2)H.$$

Prove that $\circ$ is well defined and that $(G/H, \circ)$ is a group.

Proof. To prove that $\circ$ is well defined, suppose that $k_1 \in g_1H$ and $k_2 \in g_2H$. We must show that

$$k_1H \circ k_2H = g_1H \circ g_2H.$$

Since equivalence classes partition a set, it is enough to show that $(k_1k_2) \in (g_1g_2)H$. Since $k_1 \in g_1H$, there exists $h_1 \in H$ so that $k_1 = g_1h_1$. Similarly, there exists $h_2 \in H$ so that $k_2 = g_2h_2$. Since H is normal in G, there exists $h \in H$ so that $g_2^{-1}h_1g_2 = h$. Thus,

$$k_1k_2 = g_1h_1g_2h_2 = g_1g_2(g_2^{-1}h_1g_2)h_2 = g_1g_2(hh_2).$$

Since H is a subgroup $hh_2 \in H$. Thus, $k_1k_2 \in (g_1g_2)H$. Consequently, $\circ$ is well defined.

The associativity of $\circ$ follows immediately from the associativity of $\cdot$ (the group operation on G). The identity element of G/H is the coset H. The inverse of gH is $g^{-1}H$. Hence G/H is a group. $\square$

Definition 2.23. Let G and K be groups and let $f\colon\ G \to H$ be a function. Then f is a **homomorphism** if for all $g_1, g_2 \in G$,

$$f(g_1g_2) = f(g_1)f(g_2).$$

If f is bijective, f is said to be an **isomorphism**.

Exercise 2.24. Let $n \in \mathbb{N}$. Show that the group C_n is isomorphic to the group $\mathbb{Z}/n\mathbb{Z}$. We will often denote any group isomorphic to these groups as $\mathbb{Z}_n$.

Definition 2.25. Let G be a group and let X be a set. A **action** of G upon X is a function $*\colon\ G \times X \to X$ such that

(a) If e is the identity element of G, then for all $x \in X$, $e * x = x$.
(b) For all $g, h \in G$, $g * (h * x) = (gh) * x$.

If X is a topological space, we will often require, for each g, the map $x \to gx$ to be continuous. If G is a topological group and if X is a topological space, we will require $*$ to be a continuous function.

Exercise 2.26. For the following groups G, sets X, and functions $*$, show that $*$ is an action of G on X.

(a) Let $G = \mathbb{Z}$, $X = \mathbb{R}$, and $g * x = g + x$.
(b) $G = GL_2(\mathbb{R})$ and $X = \mathbb{R}^2$. The function $*$ is matrix-vector multiplication.
(c) $G = C_n$ and X is a regular n–gon in $\mathbb{C}$ centered at the origin and with one vertex at 1. (The action should rotate X by $2\pi/n$ radians.)

Exercise 2.27. (Quotienting by a group action) Suppose that G is a group and that X is a set such that $*$ is an action of G on X. Define a relation $\sim_*$ on X by $x \sim_* y$ if and only if there exists $g \in G$ such that $g * x = y$. Show that $\sim_*$ is an equivalence relation on X. The set of equivalence classes is often denoted by $G\backslash X$.

Exercise 2.28. For each of the groups G and sets X in Exercise 2.26 describe (geometrically or otherwise) the sets $G\backslash X$.

Definition 2.29. Let $(X, \mathcal{X})$ be a topological space and let$\sim$ be an equivalence relation on X. Let $f\colon\ X \to X/\!\sim$ be the function $f(x) = [x]$. Define a set $U \subset X/\!\sim$ to be open if and only if $f^{-1}(U) \in \mathcal{X}$. The set of all open sets in $X/\!\sim$ is the **quotient topology** on $X/\!\sim$.

Exercise 2.30. Show that the quotient topology is, in fact, a topology.

Exercise 2.31. Show that the quotient topology is the largest topology on $X/\!\sim$ for which the quotient function f is continuous.

Exercise 2.32. Consider the set $I = [0, 1]$ with the usual topology. Let $A = \{0, 1\}$ and give I/A the quotient topology. Prove that I/A is homeomorphic to S^1.

Exercise 2.33. Give $I^2 = [0, 1] \times [0, 1]$ the usual topology. Define an equivalence relation on I^2 by

- $(x, y) \sim (x, y)$ for all $(x, y) \in I^2$
- $(x, 0) \sim (x, 1)$ for all $x \in I$
- $(0, y) \sim (1, y)$ for all $y \in I$.

Prove that $I^2/\!\sim$ with the quotient topology is homeomorphic to T^2.

Exercise 2.34. Use the quotient topology to give more detail or more certainty to your descriptions in Exercise 2.28.

Exercise 2.35. Let $X = \mathbb{R}^n\backslash\{0\}$. Say that $x_1 \sim x_2$ if there exists $\lambda > 0$ such that $x_1 = \lambda x_2$. Prove that $\sim$ is an equivalence relation and that $X/\sim$ with the quotient topology is homeomorphic to S^{n-1}.

Exercise 2.36. Let $X = \mathbb{R}^n\backslash\{0\}$. Say that $x_1 \sim x_2$ if there exists a real number $\lambda \neq 0$ such that $x_1 = \lambda x_2$. It is a fact that $\sim$ is an equivalence relation. Let $\mathbb{P}^{n-1} = X/\sim$ with the quotient topology. Prove that there is a surjective function $f\colon\ S^{n-1} \to \mathbb{P}^{n-1}$ such that f is a local homeomorphism. That is, for each point $x \in S^{n-1}$ there is an open set $U \subset S^{n-1}$ such that $f|_U$ is a homeomorphism onto $f(U)$.

Exercise 2.37. Show that S^1 and $\mathbb{P}^1$ are homeomorphic.

It is a fact that for $n > 1$, S^n and $\mathbb{P}^n$ are not homeomorphic.

Exercise 2.38. Define an action of the group $\mathbb{Z}^2$ on the topological space $\mathbb{R}^2$ by

$$(n, m) * (x, y) = (x + n, y + m).$$

Show that $\mathbb{Z}^2 \backslash \mathbb{R}^2$ is homeomorphic to T^2.

Exercise 2.39. Let $X = B^2 \times \{0\}$ and $Y = B^2 \times \{1\}$. Notice that X and Y are both homeomorphic to B^2. Let $A = S^1 \times \{0\} \subset X$ and $B = S^1 \times \{1\} \subset Y$. Define $f\colon\ A \to B$ by $f(s, 0) = (s, 1)$. Prove that $Y \cup_f X$ with the quotient topology is homeomorphic to S^2. Generalize, if you can, this result to higher dimensions.

3. Topological Properties

3.1. Separating points.

Definition 3.1. A topological space $(X, \mathcal{X})$ is **Hausdorff** (or T_2) if for every $x, y \in X$ with $x \neq y$ there exist disjoint open sets U_x and U_y containing x and y respectively.

Exercise 3.2. Prove that every metric space is Hausdorff.

Exercise 3.3. Let $X = (\mathbb{R} \backslash \{0\}) \cup \{a, b\}$. Define a base for a topology on X as follows. Declare the following sets to be open intervals:

(a) (x, y) such that $x < 0$ and $y \leq 0$
(b) (x, y) such that $x \geq 0$ and $y > 0$
(c) $(x, 0) \cup \{a\} \cup (0, y)$ for $x < 0$ and $y > 0$.
(d) $(x, 0) \cup \{b\} \cup (0, y)$ for $x < 0$ and $y > 0$.

Let $\mathcal{X}$ be the topology on X generated by these open intervals.

(a) Show that $(X, \mathcal{X})$ is not Hausdorff.
(b) Show that $X/\{a, b\}$ is homeomorphic to $\mathbb{R}$ and is, therefore, Hausdorff.

Exercise 3.4. Define an equivalence relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

Prove that $\mathbb{R}/\!\sim$ is non-Hausdorff.

Exercise 3.5. Let $(X, \mathcal{X})$ be a topological space. Let $A \subset X$. The **closure** of A, denoted $\overline{A}$ is the smallest closed set containing A. A set $A \subset X$ is **dense** in X if $\overline{A} = X$.

(a) Prove that for any $A \subset X$, the set $\overline{A}$ exists.

(b) Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 3.6. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Let $A \subset X$ be a dense subset of X. Suppose that $f\colon X \to Y$ is a homeomorphism. Prove that $f(A)$ is dense in Y.

Exercise 3.7. Let $(X, \mathcal{X})$ be a topological space. Define an equivalence relation $\sim$ on X by $x \sim y$ if there do not exist disjoint open sets U_x and U_y containing x and y respectively.

(a) Prove that $\sim$ is an equivalence relation.
(b) Prove that $X/\sim$ with the quotient topology is Hausdorff.
(c) Construct a topological space X such that $X/\sim$ is homeomorphic to $\mathbb{R}$ and such that the set

$$A = \{[x] : [x] \text{ contains more than one element } \}$$

is dense in $X/\sim$. (Hint: It is possible to make $A = X/\sim$.)

3.2. **Connectedness.**

Definition 3.8. Let $(X, \mathcal{X})$ be a topological space. X is **connected** if the only sets which are both open and closed are X and $\varnothing$. Suppose that $A \subset X$ is a non-empty set which is both open and closed. If, in the subspace topology, A is connected then A is a **connected component** of X.

Exercise 3.9. Prove that $(X, \mathcal{X})$ is not connected if and only if there exist nonempty disjoint open sets $A, B \subset X$ such that $X = A \cup B$. Notice that A and B are both open and closed.

Exercise 3.10. Give the set $N = \{1, 2, \ldots, n\}$ the discrete topology. Prove that the number of connected components of a topological space X is at least n if and only if there exists a continuous surjection of X onto N.

Exercise 3.11. (a) Prove that $I = [0, 1]$ is connected. (Hint use Exercise 3.10 and a theorem from Calculus.)
(b) Prove that if X and Y are connected topological spaces then $X \times Y$ is connected. Conclude that B^n is connected for $n \geq 1$.
(c) Prove that if X and Y are homeomorphic topological spaces and if X is connected then so is Y.

Exercise 3.12. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Prove that if $(X, \mathcal{X})$ is connected and if there exists a surjective continuous function from X to Y then Y is connected. Conclude that if X is connected and if $\sim$ is an equivalence relation on X then $X/\sim$ is connected.

Exercise 3.13. Prove that S^1 and T^n for $n \geq 2$ are connected.

Definition 3.14. Let $(X, \mathcal{X})$ be a topological space. A **path** in X is a continuous function $p\colon I \to X$. We say that p is a path from $p(0)$ to $p(1)$. A space $(X, \mathcal{X})$ is **path-connected** if for all $x, y \in X$ there exists a path from x to y.

Exercise 3.15. Prove that if X is path-connected then it is connected.

Exercise 3.16. Prove that $(0, 1)$ and $\mathbb{R}$ are connected.

Exercise 3.17. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Prove the following.

(a) If X is connected and X and Y are homeomorphic then Y is connected.
(b) Suppose that $f\colon X \to Y$ is a homeomorphism. Let $x \in X$. Give $X - \{x\}$ and $Y - \{f(x)\}$ the subspace topologies. Prove that $X - \{x\}$ is homeomorphic to $Y - \{y\}$.
(c) Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.
(d) Prove that I is not homeomorphic to $(0, 1)$.
(e) Prove that I is not homeomorphic to S^1.
(f) Prove that the comb space (Exercise 1.13) is not homeomorphic to the upper half space with the usual topology.

Definition 3.18. $(X, \mathcal{X})$ is **locally connected** if for each point $x \in X$ and each open set $U \subset X$ such that $x \in U$, there exists an open set $V \subset U$ such that $x \in V$ and V is connected.

Exercise 3.19. (The topologists' sine curve) Let S be the graph of $y = \sin(1/x)$ for $x > 0$ in $\mathbb{R}^2$. Let

$$J = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$$

Define $\Gamma = S \cup J$. Γ (with the subspace topology) is the topologists' sine curve.

(a) Show that Γ is connected.
(b) Show that Γ is not locally connected.
(c) Show that Γ is not path-connected.
(d) Modify the construction of Γ to produce an example of a space which is path connected but not locally connected.

3.3. **Covers and Compactness.**

Definition 3.20. Let $(X, \mathcal{X})$ be a topological space. A **cover** of X is a collection of open sets $\{U_\alpha : \alpha \in A\}$ (for some indexing set A) such that $\cup U_\alpha = X$. Sometimes this is called an **open cover**.

Definition 3.21. A topological space $(X, \mathcal{X})$ is **second countable** if and only if there is a countable base for the topology.

Exercise 3.22. Show that a metric space (X, d) is second countable if every open cover has a countable subcover.

Exercise 3.23.
(a) Show that $\mathbb{R}^n$ with the usual topology is second countable.
(b) Show that if $(X, \mathcal{X})$ is second countable and if $A \subset X$ is given the subspace topology then A is second countable.
(c) Prove that if (X, d) is a metric space and if X contains a countable dense subset, then X is second countable.
(d) Prove that if $(X, \mathcal{X})$ is second countable then X contains a countable dense subset.

Exercise 3.24. Let $X = \mathbb{R} \times \mathbb{R}$. Define $(a, b) < (x, y)$ if $a < x$ or if $a = x$ and $b < y$. (This is called the **lexicographic order** on X.) Define the interval $(\mathbf{a}, \infty)$ for $\mathbf{a} \in X$ to be

$$(\mathbf{a}, \infty) = \{\mathbf{x} \in X : \mathbf{a} < \mathbf{x}\}$$

Similarly define

$$(-\infty, \mathbf{a}) = \{\mathbf{x} \in X : \mathbf{x} < \mathbf{a}\}$$

Let $\mathcal{T}$ be the topology on $\mathcal{X}$ generated by these intervals. This topology is called the **order topology** on X.

(a) Prove that $(X, \mathcal{T})$ is connected and Hausdorff.
(b) Prove that $(X, \mathcal{T})$ is not homeomorphic to $\mathbb{R}^2$ with the usual topology.
(c) (Hard?) Is $(X, \mathcal{T})$ second countable?

Definition 3.25. Let $(X, \mathcal{X})$ be a topological space. X is **compact** if every cover of X has a finite subcover.

Exercise 3.26. Prove that $\mathbb{R}^n$ is not compact for $n \geq 1$.

Exercise 3.27. Let $(X, \mathcal{X})$ be a Hausdorff topological space and suppose that $A \subset X$ is a compact subspace. Prove that A is closed.

Exercise 3.28. Let X and Y be compact topological spaces. Prove the following:

(a) The projection of $X \times Y$ onto Y is a closed map. That is, the image of a closed set is a closed set. (This does not use the fact that Y is compact.)
(b) The projection of $X \times Y$ onto Y is a proper map. That is, the preimage of a compact set is compact. (This does not use the fact that Y is compact.)
(c) $X \times Y$ is compact.

Exercise 3.29. Let X and Y be topological spaces. Suppose that X is compact and $f\colon X \to Y$ is continuous. Show that $f(X)$ is compact (with the subspace topology in Y).

Exercise 3.30. Prove that if $A \subset X$ is closed and if X is compact, then A is compact.

Exercise 3.31. Prove that if X is compact and if Y is Hausdorff and if $f\colon X \to Y$ is a continuous bijection then f is a homeomorphism.

The space $\mathbb{R}$ is characterized by the fact that ever bounded subset of $\mathbb{R}$ has a least upperbound.

Theorem 3.32. The unit interval $I = [0, 1]$ is compact.

Proof. (See Bredon's Topology and Geometry.) Let $\mathcal{U}$ be an open covering of I. Let

$$S = \{s \in I : [0, s] \text{ is covered by a finite subcollection of } \mathcal{U}\}.$$

Let b be the least upperbound of S. Obviously, either $S = [0, b]$ or $S = [0, b)$ (proof?). Suppose that $S = [0, b)$. Let $\mathcal{U}_1$ be a finite subcover of $\mathcal{U}$ which covers S. Let U be any open set in $\mathcal{U}$ which contains b. Then $\mathcal{U}_1 \cup \{U\}$ is a finite subset of $\mathcal{U}$ which covers $[0, b]$. Hence, $b \in S$. Thus, $S = [0, b]$.

If $b < 1$ a similar argument (what is it?) shows that we encounter a contradiction. Hence, $b = 1$ and a finite subcover of $\mathcal{U}$ covers I. $\square$

Exercise 3.33. (Heine-Borel) Suppose that $X \subset \mathbb{R}^n$ is a closed and bounded subset. Then X is compact (with the subspace topology).

Exercise 3.34. (Extreme Value Theorem) Let X be a compact topological space and suppose that $f\colon X \to \mathbb{R}$ is continuous. Then there is $x \in X$ such that for all $y \in X$, $f(x) \geq f(y)$. The number $f(x)$ is the **global maximum** of f. Similarly, show that f attains its global minimum.

Exercise 3.35. Prove that S^n is compact. Use this to show that, at a fixed moment in time, someplace on earth has a temperature that is equal to or larger than the temperature at every other place on earth.

Exercise 3.36. Prove that S^n and $\mathbb{R}^m$ are not homeomorphic for any n, m.

Exercise 3.37. Suppose that X is a Hausdorff space with infinitely many points. Let $x \in X$. Prove that $X \backslash \{x\}$ is not compact.

Exercise 3.38. Prove that the middle thirds Cantor set is compact.

Exercise 3.39. (One-point compactification) Let $(X, \mathcal{X})$ be a locally compact Hausdorff topological space. Let $*$ be a point not in X. Let $\widehat{X} = X \cup \{*\}$. Define a topology $\mathcal{T}$ on $\widehat{X}$ by declaring a set to be open if it is in $\mathcal{X}$ or if it is $\widehat{X} - C$ for some compact set $C \subset X$. Prove that $(\widehat{X}, \mathcal{T})$ is a compact, Hausdorff space.

Exercise 3.40. Use the notation from the previous exercise. Prove that $\mathcal{T}$ is the unique topology on $\widehat{X}$ which makes $\widehat{X}$ into a compact, Hausdorff space.

Exercise 3.41. Prove that S^n is the one-point compactification of $\mathbb{R}^{n-1}$.

Exercise 3.42. Prove that the one point compactification of the rationals in $\mathbb{R}$ is not Hausdorff.

Exercise 3.43. Prove that the one point compactification of a space $(X, \mathcal{X})$ is connected if and only if X is not compact.

Exercise 3.44. (Hawaiian earring space) Let C be the subset of $\mathbb{R}^2$ consisting of the union of circles of radius $1/n$ with center at $(0, 1/n)$. Let X_i be the vertical line in $\mathbb{R}^2$ passing through the point $(i, 0)$ for $i \in \mathbb{N}$. Let $\widehat{X}$ be the one point compactification of $\cup_{\mathbb{N}} X_i$. Prove that C and $\widehat{X}$ are homeomorphic.

Exercise 3.45. (The infinite rose) Let S_i be a copy of S^1 for each $i \in \mathbb{N}$. Let S be the disjoint union of the S_i. Let x_i be a point in S_i. Let $A = \{x_1, \ldots\}$. Let $R = S/A$. Show that R is not homeomorphic to the Hawaiian earring. (Hint: show that there is a point in R which has a neighborhood which is a tree with infinitely many branches. Show that there is no such point in the Hawaiian earring space.)

4. SEQUENCES

Definition 4.1. A **sequence** in a set X is a function $f\colon \mathbb{N} \to X$. We often write x_i instead of $f(i)$.

Definition 4.2. Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a sequence in a topological space X. A point $l \in X$ is a **limit point** of $\mathbf{x}$ if for every open set U containing l there is an $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n \in U$.

Exercise 4.3.

(a) Prove that if X has the indiscrete topology then every point of X is a limit point of every sequence in X.
(b) Prove that if X has the discrete topology then the only sequences that have limit points are sequences which are eventually constant.
(c) Prove that if a sequence in a Hausdorff space has a limit point then that limit point is unique.

Definition 4.4. Let X be a topological space and let $A \subset X$. A **limit point** of A is a point $x \in X$ such that for every open set containing x, $U \cap A$ contains a point other than x.

Exercise 4.5. Let X be a metric space and let $A \subset X$.

(a) Prove that if x is a limit point of A then $x \in \overline{A}$.
(b) Let L be the set of limit points of A. Prove that $\overline{A} = A \cup L$.

Exercise 4.6. Prove that every point of the middle thirds Cantor set C is a limit point. Hence, $\overline{C} = C$.

Exercise 4.7. Let $A \subset X$ be a subset of a metric space.

(a) Prove that every limit point of A is the limit of some sequence in A.
(b) Give an example of X and A such that there is a convergent sequence in A which does not converge to a limit point of X.
(c) Prove that any such sequence (in part (b)) must eventually be constant.

Definition 4.8. A function $f\colon\ X \to Y$ is said to be **sequentially continuous** if whenever a sequence $\{x_i\} \subset X$ converges to $L \in X$, the sequence $\{f(x_i)\} \subset Y$ converges to $f(L) \in Y$.

Exercise 4.9. Let X be a metric space and Y a Hausdorff space and $f\colon\ X \to Y$ a function. Prove that f is continuous if and only if it is sequentially continuous.

Exercise 4.10. (Closed sets contain limit points) Let $(X, \mathcal{X})$ be a Hausdorff space. Let $A \subset X$. Prove that A is closed if and only if whenever a sequence $\{a_i\} \subset A$ converges to a point $L \in X$, $L \in A$.

Definition 4.11. A space $(X, \mathcal{X})$ is **sequentially compact** if every sequence in X contains a convergent subsequence.

Exercise 4.12. Prove that if a metric space is compact then it is sequentially compact.

Definition 4.13. Let (X, d) be a metric space and let $\{x_i\} \subset X$ be a sequence. Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) < \epsilon$. Then the sequence $\{x_i\}$ is a **Cauchy sequence**. If every Cauchy sequence in X converges then (X, d) is a **complete** metric space.

Exercise 4.14.
(a) Prove that if (X, d) is a compact metric space then (X, d) is complete.
(b) Give an example of a metric space which is complete but noncompact.

Definition 4.15. A metric space (X, d) is **totally bounded** if for each $\epsilon > 0$ there exist finitely many points $\{x_i\} \subset X$ such that $\{B_\epsilon(x_i)\}$ is a cover of X.

Exercise 4.16. Let (X, d) be a metric metric space. Prove that the following are equivalent:

(a) X is compact
(b) X is sequentially compact
(c) X is complete and totally bounded

Definition 4.17. If $A \subset X$ is a subset of a metric space (X, d), then the **diameter** of A is

$$\operatorname{diam}(A) = \sup\{d(p, q) : p, q \in A\}$$

Exercise 4.18. Prove that a compact metric space has finite diameter.

Exercise 4.19. Give an example of two homeomorphic metric spaces such that one of them has finite diameter and the other infinite diameter.

Exercise 4.20. (Lebesgue Covering Lemma[1]) Let (X, d) be a compact metric space and let $\{U_\alpha\}$ be an open covering of X. Then there exists $\delta > 0$ such that for each set $A \subset X$ with $\operatorname{diam}(A) < \delta$ there exists α such that $A \subset U_\alpha$. The number δ is the **Lebesgue number** for $\{U_\alpha\}$.

Exercise 4.21. (Contraction maps[2]) Let (X, d) be a complete metric space. Let $f\colon X \to X$. Suppose that there exists $0 \leq \lambda < 1$ such that for every $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

[1]The statement comes from Bredon's Topology and Geometry

[2]Taken from Browder's Mathematical Analysis

Prove that there is a unique $x \in X$ such that $f(x) = x$. The point x is called a **fixed point** of f.