Lectures on Geometric Topology

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1. Euler Characteristic of Finite Graphs

Exercise 1.1. Draw a graph G on a piece of paper and count the number of vertices $V = V(G)$, edges $E = E(G)$ and faces $F = F(G)$. When counting the faces include the exterior of the graph as a face.

(a) What is $V - E + F$? Compare this number with the number your neighbors obtained.
(b) Make a conjecture about $V - E + F$ for a **connected** graph.
(c) Make a conjecture about $V - E + F$ for a **disconnected** graph.

Theorem 1.2. For a finite, connected graph G in the plane with $V > 0$ vertices, E edges, and F faces (including the exterior)

$$V - E + F = 2$$

Before proving the theorem, we first prove it for a special class of graphs.

Definition 1.3. A **tree** is a connected graph with no loops.

Lemma 1.4. Suppose that T is a tree with $V > 0$ vertices and E edges. Then $V - E = 1$.

Proof. We prove this by induction on E. Suppose that $E(T) = 0$. Then, since T is connected, $V(T) = 1$. Clearly, $V(T) - E(T) = 1$. Now suppose that the statement is true for all trees T' with fewer than $E(T)$ edges. Since T is finite, there exists an edge e such that one endpoint v is not the endpoint of any other edge. (Proof?) Let T' be obtained from T by removing the edge e and the vertex v, but not the other endpoint of e. Notice that $V(T') = V(T) - 1$ and $E(T') = E(T) - 1$. Since v was not the endpoint of any other edge, T' is connected. Furthermore, T' is a tree. (Proof?) Thus, by the inductive hypothesis, $V(T') - E(T') = 1$. Hence, $V(T) - E(T) = (V(T') + 1) - (E(T') + 1) = 1$. □

Definition 1.5. A **maximal tree** T in a graph G is a subgraph of G which is a tree and which contains all the vertices of G.

We now prove the theorem.

Proof of Theorem 1.2. Let G be a graph in $\mathbb{R}^2$ with $V(G)$ vertices, $E(G)$ edges, and $F(G)$ faces. Choose a maximal tree $T \subset G$ (why does one exist?). Let $n(G,T)$ be the number of edges of $G-T$. If $n(G,T)=0$, then G is a tree and so by Lemma 1.2, $V(G)-E(G)=1$. If G is a tree, the only face is the exterior face and so $F(G)=1$. Hence, if $n(G,T)=0$, $V(G)-E(G)+F(G)=2$.

Let G' be a graph and $T' \subset G'$ be a maximal tree. Suppose that the theorem is true for G' if $n(G',T') < n(G,T)$. Let e be an edge of $G-T$ and let G' be the graph obtained by removing e from G. Then G' is a connected graph (proof?) and $T'=T \subset G'$ is a maximal tree for G' (proof?). Furthermore, $n(G',T')=n(G,T)-1$. Thus, by the inductive hypothesis,

$$V(G')-E(G')+F(G')=2.$$

Now, G is obtained from G' by attaching the edge e. The edge e lies in some face of G' and therefore (?) must cut it into two faces. Thus, $F(G)=F(G')+1$. Hence,

$$V(G)-E(G)+F(G)=V(G')-(E(G')+1)+(F(G')+1)=2.$$

□

We now consider graphs on the 2–sphere S^2. For such a graph G, let $V(G)$, $E(G)$, and $F(G)$ denote the number of vertices, edges, and faces as before.

Lemma 1.6. (Stereographic Projection) Let $G \subset S^2$ be a finite connected graph. Then there is a connected graph $G' \subset \mathbb{R}^2$ such that $V(G')=V(G)$, $E(G')=E(G)$ and $F(G')=F(G)$.

Proof. Let $N \in S^2$ be a point not on G. (Does such a point exist?) Rotate and translate S^2 in $\mathbb{R}^3$ so that the center of S^2 is $(0,0,1) \in \mathbb{R}^3$ and the point $N=(0,0,2)$. For each point $x \in S^2 - \{N\}$ define a point $y \in \mathbb{R}^2$ as follows. Draw the line L_x through the points N and x. The line L_x intersects the xy–plane $\mathbb{R}^2$ in a unique point. Call that point y. We have defined a function

$$\texttt{stereo}\colon S^2 - \{N\} \to \mathbb{R}^2.$$

Let $G' = \texttt{stereo}(G)$. It is easy to see that G' satisfies the conclusion of the lemma. □

Corollary 1.7. For a finite, connected graph G on S^2, $V(G)-E(G)+F(G)=2$.

This corollary has a very nice geometric consequence.

Definition 1.8. A **regular polyhedron** is a polyhedron all of whose faces are regular n–gons and such that every vertex has the same number of faces incident to it. A convex regular polyhedron is called a **Platonic solid**.

Theorem 1.9. (Platonic solids) There are at most 5 Platonic solids.

Proof. Let P be a platonic solid with faces that are regular n–gons and with s faces around each vertex. The vertices and edges of P form a graph G. Let V be the number of vertices, E the number of edges, and F the number of faces. Notice that:

$$\begin{aligned} E &= nF/2 \qquad\qquad \text{and} \\ V &= nF/s. \end{aligned}$$

Inscribe P in a sphere S^2 of radius 1. (Can this be done?) Use radial projection to send G to a graph G' on S^2. Notice that G' is connected and that $V(G') = V$, $E(G') = E$, and $F(G') = F$. Hence, $V - E + F = 2$ by Corollary 1.7. Consequently,

$$F\left(\frac{n}{s} - \frac{n}{2} + 1\right) = 2.$$

This can be rewritten as

$$F\left(\frac{2(n+s) - ns}{2s}\right) = 2$$

Recall that F, n, and s are all positive integers. Since the faces of the platonic solid are rigid n–gons, $F \geq 3$. Since the interior angle of a regular n–gon is less than 180°, $s \geq 3$ and $n \geq 3$.

Notice that $2(n+s) - ns > 0$. This means that $n < 2s/(s-2)$. If $s = 7$, then $2s/(s-2) = 2.8$. The function $s \to 2s/(s-2)$ is a decreasing function and so for $s \geq 7$, $n < 3$. Similarly, if $n \geq 7$, $s < 3$. Thus, both n and s are between 3 and 6. Of the possibilities for (n,s) the only ones that make $\frac{2(n+s)-ns}{2s}$ positive are: $(3,3)$, $(4,3)$, $(5,3)$, $(3,4)$, and $(3,5)$. In particular, there are just five possibilities for n and s given these constraints. $\square$

The cube, tetrahedron, octahedron, dodecahedron, and icosahedron are five Platonic solids. Using the previous theorem, it is possible to show that they are the only five.