2. The polygonal Jordan Curve Theorem

A simple closed curve C in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ such that, with the subspace topology, it is homeomorphic to S^1. It is **polygonal** if there are a finite number of points $\mathscr{V} \subset C$ such that $C - \mathscr{V}$ is a collection of straight line segments. Notice that the number of such line segments must be finite.

Theorem 2.1 (Jordan Curve Theorem)**.** If C is a simply closed polygonal curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \backslash C$ has exactly two connected components. These are, in fact, path-connected components.

Proof. The proof proceeds in two stages: first, we show that the number of connected components is no more than two and second, we show it is not less than two.

Step 1: Let $x \in C \backslash \mathscr{V}$. Since the number of edges of $C \backslash \mathscr{V}$ is finite, there is a disc $B = B(x, \delta)$ centered at x such that B is disjoint from $\mathscr{V}$ and $B \cap C$ is a (connected) line segment. (This uses the fact that the distance from x to a compact set attains its minimum.) It is clear that $B \backslash C$ has two components. Let $y \in \mathbb{R}^2 \backslash C$. Since $\mathbb{R}^2$ is path connected, there is a path from y to C which has interior disjoint from C. Since C is polygonal, rather than having the path go from y to C, the path can go from y to a line "close to" C. "Close to", means that there is a disc between the line and C whose interior is disjoint from C. See Figure 1. Following lines near to C, the path follows the course of C (without ever intersecting it) until it enters B. Thus, each connected component of $\mathbb{R}^2 \backslash C$ contains a component of $B \backslash C$. Hence, there are at most two connected components. Notice that the argument implies that these are path connected components.

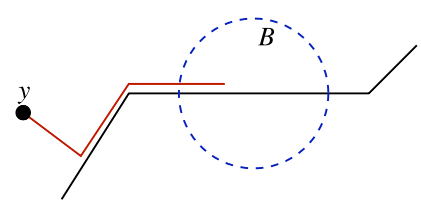

FIGURE 1. A path entering B on one side of C.

Step 2: To show that $\mathbb{R}^2 \backslash C$ has not fewer than two connected components, we will create a continuous surjection $f\colon \mathbb{R}^2 \backslash C \to \{0, 1\}$ where $\{0, 1\}$ has the discrete topology. Since $\{0, 1\}$ has two elements, this is enough to guarantee that $\mathbb{R}^2 \backslash C$ has at least two connected components.

Let $x \in \mathbb{R}^2 \backslash C$ and let γ be a ray extending from x with angle θ. The intersection $x \cap \gamma$ consists of line segments and points not on line segments contained in $x \cap \gamma$. Call these latter points, “individual intersection points”. Let $\mathscr{I}(\theta)$ be the set of these line segments and individual intersection points. See Figure 2. If $p \in \mathscr{I}(\theta)$ Let $i(p,\theta)$ be 0 if the edges adjacent to y are on the same side of γ and let $i(p,\theta)$ be 1 if they are on opposite sides.

$$f(x,\theta) = \sum_{p \in \mathscr{I}(\theta)} i(p,\theta) (\text{mod } 2).$$

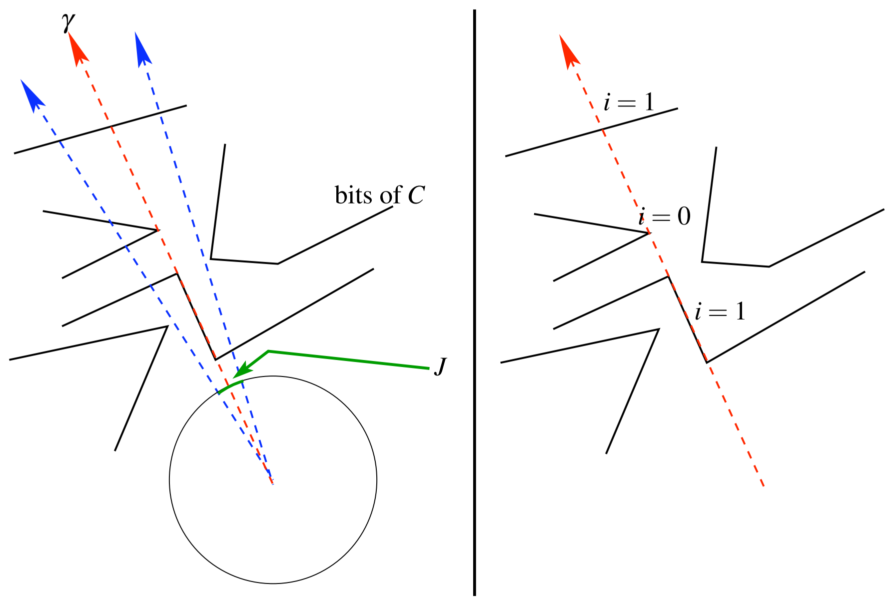

FIGURE 2. Calculating f.

Lemma 2.2. Let x be fixed. For any two angles θ, θ', $f(x,\theta) = f(x,\theta')$.

Proof. We prove that $f_x = f(x,\cdot)\colon S^1 \to \{0,1\}$ is continuous. Since S^1 is connected, this implies that f_x is constant. Suppose that $\theta \in S^1$, we will show that there is an open set containing θ on which f_x is constant. Let γ be the ray with angle θ.

Since $\mathscr{V}$ is finite, there is an open interval J on S^1 containing θ such that for all rays ψ with angle ϕ in J, either $\psi = \gamma$ or $\psi \cap \mathscr{V} = \varnothing$. See Figure 2. If $i(p,\theta) = 0$, then for all ϕ to one side of θ in J there are two points of intersection of ψ with the edges of C adjacent to p. For all ϕ to the other side of θ in J, ψ does not intersect at all the edges of C adjacent to p. Similarly, if $i(p,\theta) = 1$, then to calculate $f(x,\psi)$ there is one edge adjacent to p which intersects ψ. Thus, because we work modulo 2, the change of intersection at p is unchanged. Since this is true for all p in $\mathscr{I}(\theta)$, $\mathscr{I}(\theta) = \mathscr{I}(\psi)$ for all $\psi \in J$. $\square$

Consequently, define $f(x)$ to be $f(x,\theta)$ for any angle $\theta \in S^1$.

Lemma 2.3. The function $f\colon\ \mathbb{R}^2 \backslash C \to \{0,1\}$ is continuous.

Proof. Once again, we will show that for all x in $\mathbb{R}^2\backslash C$ there is a ball centered at x on which f is constant. Let $x \in \mathbb{R}^2\backslash C$. Since C is compact, there is a ball D centered at x contained in $\mathbb{R}^2\backslash C$. Let $y \in D$. Since D is convex. There is a line segment e joining x and y which is contained in D and is, therefore, disjoint from C. Let γ_y be the ray emanating from y containing e and let γ_x be the ray emanating from x contained in γ_y. Let θ_y and θ_x be the corresponding angles. Since e is disjoint from C,

$$f(x,\theta_x) = f(y,\theta_y).$$

Since $f(x,\theta_x)$ and $f(y,\theta_y)$ are independent of θ_x and θ_y, the result follows immediately. $\square$

It remains to show that f is surjective. Let x and y be in $B\backslash C$ and be on opposite sides of C. Let γ_x be the ray based at x and passing through y. Since B is convex, $\gamma_x \cap C$ intersects C exactly once. Let γ_y be the ray based at y and contained in γ_x. Then $\gamma_x \cap C$ has one more component than does $\gamma_y \cap C$. This component is a point in the interior of an edge of C and so $f(x) = f(y) + 1$(modulo 2). Thus, f is surjective.

Therefore, $\mathbb{R}^2\backslash C$ has exactly two components. $\square$

The Schönflies Theorem says that one component of $\mathbb{R}^2\backslash C$ is homeomorphic to a disc and the other is homemorphic to a disc minus a point. This is harder to prove.