## Final Exam

- This exam is due in Professor Taylor's office on Saturday.
- You may not discuss this exam with anybody other than the professor.
- You may use your textbook, the point-set topology notes, your own course notes, and any other handouts distributed in class or placed on the course webpage.
- Other than those resources listed above, you may not use any other resources in completing this exam. This includes other topology texts and online sources.

## 1. Point-Set Topology

(1) Before beginning this problem, you may wish to read about the one-point compactification of a space on page 90 of your text. Let $(X, d)$ be a connected non-compact metric space. Let $x_0 \in X$ and suppose that $X$ has the property that $B_r = \{y \in X : d(x_0, y) \leq r\}$ is compact for every $r > 0$. Let $W_r = \{y \in X : d(x, y) \geq r\}$. Notice that $W_r$ is closed. Let $\{r_n\}$ be an sequence of real numbers so that $\lim_{n\to\infty} r_n = \infty$. For each $r_n$ let $U(r_n)$ be a non-compact connected component of $W_{r_n}$. The sequence $\{U(r_n)\}$ is called a **pre-end** of $X$ if

$$U(r_1) \supset U(r_2) \supset U(r_3) \dots$$

Suppose that $\{r_n\}$ and $\{s_n\}$ are sequences so that $\{U(r_n)\}$ and $\{U(s_n)\}$ are pre-ends of $X$. Define $\{U(r_n)\} \sim \{U(s_n)\}$ iff for every $m$ there exists $n$ with $U(s_m) \supset U(r_n)$.

We will also need the following definition. Define

$$\mathring{U}(r_n) = U(r_n) \cap \{y \in X : d(x_0, y) > r_n\}.$$

Notice that although $U(r_n)$ may not be open, $\mathring{U}(r_n)$ is open.

(a) Give an example of a pre-end of the metric space $\mathbb{R}$ with the usual metric.

(b) Prove that $\sim$ is an equivalence relation on the set of pre-ends. Define an **end** of $X$ to be an equivalence class of pre-ends.

(c) Explain why $\mathbb{R}^n$ for $n \geq 2$ has exactly one end and why $\mathbb{R}$ has two ends.

Let $J$ be the collection of ends of $X$. We will now think of $J$ as a collection of points which are not in $X$ and we will topologize $\widehat{X} = X \cup J$ by defining a generating set for the topology. Let $\mathcal{B}$ consist of sets $V$ such that either $V \subset X$ and $V$ is open or $V = \mathring{U}(r_n) \cup \{u\}$ for some $u \in J$ and some $U(r_n)$ in a pre-end representing the equivalence class $u$. Give $\widehat{X}$ the topology generated by $\mathcal{B}$.

(d) Assume that for all $x \in X$, there exists $\epsilon > 0$ so that $\{y \in X : d(x, y) \leq \epsilon\}$ is compact. Prove that $\widehat{X}$ is Hausdorff.

(e) Suppose that $X$ has finitely many ends. Let $\mathcal{V}$ be an open cover of $\widehat{X}$ by sets in $\mathcal{B}$. Prove that $\mathcal{V}$ has a finite subcover of $\widehat{X}$. (In fact, $\widehat{X}$ is compact, but answering this problem is easier than proving that fact.)

(f) Bonus: Give $\widehat{X}/J$ the quotient topology. Prove that this is homeomorphic to the one-point compactification of $X$.

## 2. Problems Concerning Major Theorems Proved in Class

Do **three** of the following 5 problems.

(1) Let $p = (1, 0) \in S^1 \subset \mathbb{R}^2$. In a homework exercise, you proved that given a loop $f\colon\ I \to S^1$, based at $p$, $f$ can be homotoped (without moving the basepoint) to either a constant map or to a map which is monotonic on $S^1$. In order to show that $\pi(S^1, p)$ is isomorphic to $\mathbb{Z}$ it remains to show that no monotonic based loop is homotopic to a constant map. In this problem you will use Sperner's lemma to partially achieve that goal.

Prove the following:

Let $f\colon\ I \to S^1$ be the based loop which travels monotonically around $S^1$ exactly once in a counterclockwise direction. (That is, $f(s) = (\cos(2\pi s), \sin(2\pi s))$. Let $f^k$ denote $f \cdot f \cdot \ldots f$ ($k$ times). Prove that if $k$ is odd, then $f^k$ is not homotopic to a constant map by a basepoint preserving homotopy.

Hint: Suppose that such a homotopy $F\colon\ I \times I \to S^1$ exists. Give $S^1$ the triangulation with 3 vertices. Think of $I \times I$ as a square. Triangulate $I \times I$ so that there are no vertices on the interiors of the side or top edges and so that there are $3k$ vertices on the bottom edge. Use the strong version of the simplicial approximation theorem (Theorem 3.18 in the text) make $F$ "nice" with respect to the triangulations. Call the vertices of $S^1$, $w_0$, $w_1$, and $w_2$. Let $v \in I \times I$ be a vertex. Give $v$ the label $i$ if $F(v) = w_i$. Then use Theorem 2.25.

(2) In class we proved the Polygonal Jordan Curve Theorem. Recall the proof. Let $R \subset \mathbb{R}^2$ be the square $[0, 1] \times [0, 1]$. Let $T^2 = R/\!\sim$ be the torus resulting from gluing opposite sides of $R$. Let $q\colon\ R \to T^2$ be the quotient map.

(a) Find a non-intersecting polygonal curve $P \subset R$ so that $q(P) \subset T^2$ is a closed curve which doesn't intersect itself and is such that $T^2 - q(P)$ has exactly one connected component.

(b) Give a careful explanation of why the proof of the Polygonal Jordan Curve Theorem does **not** also show: for every polygonal curve $P \subset R$ with $q(P)$ a non-self-intersecting closed curve, $T^2 - q(P)$ has exactly two connected components. (In other words, where does the proof of the polygonal Jordan Curve Theorem fail if we try to use it to study curves on a torus?)

(c) Outline a proof of the following theorem: Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^3$ with each simplex of $\mathcal{T}$ a geometric simplex (i.e. the convex hull of affinely independent points). Suppose that

$S \subset \mathbb{R}^3$ is the union of 2-simplices from $\mathcal{T}$ and suppose that $S$ is homeomorphic to a connected orientable surface. (In fact, it is impossible for $S$ to be homeomorphic to a non-orientable surface.) Then $\mathbb{R}^3 - S$ has exactly two connected components.

(3) Use the Invariance of Dimension Theorem (Theorem 2.8) to prove that if $M$ is an $m$-manifold without boundary and if $N$ is an $n$-manifold without boundary and if $M$ is homeomorphic to $N$ then $m = n$. (Recall that by the definition of manifold, if $x \in M$ then there is an open set containing $x$ which is homeomorphic to an open set in $\mathbb{R}^m$.)

(4) In class we proved the Brouwer Fixed Point Theorem. In this problem you will explore what happens when the hypotheses are relaxed.
   - (a) Let $A^2$ denote the annulus. Find a continuous function $f\colon A^2 \to A^2$ which has no fixed points.
   - (b) Find a continuous function $f\colon S^2 \to S^2$ which has no fixed points.
   - (c) Find a non-continuous function $f\colon D^2 \to D^2$ which has no fixed points. ($D^2$ denotes the closed unit disc.)
   - (d) Let $\mathring{D}$ denote the open unit disc in $\mathbb{R}^2$. Find a continuous function $f\colon \mathring{D} \to \mathring{D}$ without fixed points. (Hint: Recall that $\mathring{D}$ is homeomorphic to $\mathbb{R}^2$.)

(5) Consider the surface $S$ pictured below. Let $S_1$ be obtained by gluing a disc to the boundary of $S$. Let $S_2$ be obtained by gluing a Möbius band to the boundary of $S$. Use the classification of surfaces theorem to identify $S_1$ and $S_2$ as $S^2$, $nT^2$, or $m\mathbb{P}^2$ for some $n$ or $m$.

## 3. Algebraic Topology

Do **three** of the following three problems. If you elect not to do a certain problem you may still use it in the other problems, if you wish.

(1) Let $X$ and $Y$ be path connected topological spaces and let $p \in X$ and $q \in Y$. Let $r = (p, q) \in X \times Y$. Prove that $\pi_1(X \times Y, r)$ is isomorphic to $\pi_1(X, p) \times \pi_1(Y, q)$.

(2) Let $S$ be a connected closed orientable surface. Prove that if $\chi(S) < 0$ then $\pi_1(S, p)$ is not abelian. That is, there exists $a, b \in \pi_1(S, p)$ such that $ab \neq ba$. You may use the fact that a simple closed curve in a surface is homotopic to a constant loop if and only if it is the boundary of a disc in the surface.

(3) Let $S$ and $T$ each be a connected closed surface and suppose that $f\colon S \to T$ is a continous map. We say that $f$ is **incompressible** if $f_*\colon \pi_1(S, p) \to \pi_1(T, f(p))$ is injective.
   (a) Suppose that $S$ is a 2–sphere. Prove that $f$ is incompressible.
   (b) Suppose that $S$ is the projective plane and that $f$ is incompressible. Prove that $T$ is non-orientable. You may use the fact that if $T$ is an orientable surface, then no non-trivial element has finite order. That is, if $T$ is orientable and if $g$ is a based loop in $T$ which is not homotopic (by a basepoint preserving homotopy) to a constant map then $g^n = g \cdot g \cdot g \ldots \cdot g$ is not homotopic (by a basepoint preserving homotopy) to a constant map.
   (c) Suppose that $T$ is the torus, that $S$ is orientable, and that $f$ is incompressible. Prove that $S$ is either the torus or the 2–sphere

(4) Let $S$ be a surface and recall that a continuous function $f\colon S \to S$ induces a homomorphism $f_*\colon \pi_1(S, x_0) \to \pi_1(S, x_0)$. It is a fact that if two orientation preserving maps induce the same homomorphisms on $\pi_1(S, x_0)$ then the maps are homotopic. (This is not necessarily true if we work with topological spaces other than surfaces.)
   (a) Prove that if $f\colon S \to S$ is a homeomorphism then $f_*$ is an isomorphism. (Hint: Consider the map induced by the homeomorphism $f \circ f^{-1}$.)
   (b) Using the fact mentioned above, prove that a 2–sphere has exactly one orientation-preserving homeomorphism, up to homotopy.
   (c) Using the fact mentioned above, explain why orientation preserving homeomorphisms of the torus $T^2$ to itself can be classified (up to homotopy) by $2 \times 2$ matrices with integer entries whose inverse matrices also have integer entries.

(5) Let $X$ be a topological space and suppose that $A \subset X$. A retraction $r\colon X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. A deformation retraction of $X$ onto $A$ is a retraction $r\colon X \to A$ such that there exists a homotopy $R\colon X \times I \to X$ such that for all $x \in X$, $a \in A$, and $t \in I$, $R(x, 0) = x$, $R(x, 1) = r(x)$ and $R(a, t) = a$. Suppose that there exists a deformation retraction of $X$ onto $A$. Let $i\colon A \to X$ be the inclusion map. (I.e. $i(a) = a$.). Prove that

$$i_*\colon \pi_1(A, a_0) \to \pi_1(X, x_0)$$

is an isomorphism. Use this to prove that if $S$ is a surface then $S$ and $S \times I$ have the same fundamental group.