Exercise 2.14
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Theorem. The n-cube $J^n = \times_{i=1}^n [-1,1]$ is homeomorphic to the n–ball B^n.

Proof. Notice that since $\mathbb{R}^n$ has the product topology and since $J^n \subset \mathbb{R}^n$, the product topology on J^n coincides with the subspace topology on J^n. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $||x||$ denote the usual norm

$$||x|| = \Big(\sum x_i^2\Big)^{1/2}.$$

Notice that $||\cdot||\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Let d denote the usual distance function on $\mathbb{R}^n$. That is, $d(x,z) = ||x-z||$.

Define the function

$$|\cdot|\colon \mathbb{R}^n \to \mathbb{R}^n$$

by

$$|x| = (|x_1|, |x_2|, \ldots, |x_n|).$$

Lemma 1. The function $|\cdot|$ is continuous.

Proof of Lemma. Let $\varepsilon > 0$ be given. We desire to show that there exists $\delta > 0$ such that if $z \in B_\delta(x) \subset \mathbb{R}^n$ then $|z| \in B_\varepsilon(|x|) \subset \mathbb{R}^n$. Since the absolute value function from $\mathbb{R} \to \mathbb{R}$ is continuous, for each i there exists δ_i so that $|x_i - z_i| < \delta_i$ then $\Big||x_i| - |z_i|\Big| < \varepsilon/n$. Let $\delta = \min \delta_i$. Assume that $d(x,z) < \delta$. Then,

$$d(|x|,|z|) = \sqrt{\sum(|x_i| - |z_i|)^2} \leq \sum \sqrt{(|x_i| - |z_i|)^2} = \sum \Big||x_i| - |z_i|\Big| < \sum \varepsilon/n = \varepsilon.$$

Hence, if $z \in B_\delta(x)$ then $|z| \in B_\varepsilon(|x|)$. □

Now define the function $\max\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\max(x) = \max\{x_1, \ldots, x_n\}.$$

(Recall that $x = (x_1, \ldots, x_n)$.)

Lemma 2. The function max is continuous.

Proof of Lemma. Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$. We must show that there exists $\delta > 0$ so that if $d(x,z) < \delta$ then $\max(z) \in B_\varepsilon(\max x) \subset \mathbb{R}$. Suppose that

$\max(x) = x_j$. Let $\delta = \varepsilon$. Suppose $d(x,z) < \delta$. Let $z_p = \max(z)$. Notice that, for all i, $|x_i - z_i| < \delta$. This implies that, for all i,

$$x_i - \delta < z_i < x_i + \delta.$$

Now for each i,

$$z_i < x_i + \delta \leq x_j + \varepsilon$$

by the definition of δ. In particular, $z_p \leq x_j + \varepsilon$.

Also,

$$x_j - \varepsilon = x_j - \delta < z_j \leq z_p$$

since $z_p = \max(z)$. Thus,

$$x_j - \varepsilon < z_p < x_j + \varepsilon.$$

In other words, $\max(z) \in B_\varepsilon(x) \subset \mathbb{R}$. □

Define $||x||_s = \max(|x|)$. Since $||\cdot||_s$ is the composition of two continuous functions, it is continuous. (Notice that $d_{\text{sup}}(x,y) = ||x-y||_s$. We won't use this fact.)

Notice that for a fixed k the set $\{x : ||x||_s = k\}$ is a hollow n–cube centered at the origin. The distance between opposite faces is $2k$.

Define $h\colon J^n \to B^n$ by

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \in \mathbb{R}^n \\ \frac{||x||_s}{||x||}x & \text{if } x \neq 0 \end{cases}$$

Define $k\colon B^n \to J^n$ by

$$k(x) = \begin{cases} 0 & \text{if } x = 0 \in \mathbb{R}^n \\ \frac{||x||}{||x||_s}x & \text{if } x \neq 0 \end{cases}.$$

You can check that, as functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, h and k are inverse functions. Consequently, they are injective. We now show that h is surjective onto B^n. Let $y \in B^n$ and suppose that $||y|| = k$. Let $x \in J^n$ and suppose that $||x||_s = k$. Choose x so that x and y are on the same ray based at the origin. If $y = 0$ then $x = 0$. Suppose that $y \neq 0$. Then

$$h(x) = \frac{||x||_s}{||x||}x.$$

The vector $x/||x||$ is on the sphere of radius 1. Thus, $h(x)$ is on the sphere of radius $||x||_s = k$. Since x and $h(x)$ are on the same ray based at 0, and x and y are on the same ray based at 0, and $h(x)$ and y are both on the sphere of radius k, we must have $h(x) = y$. Thus, h is surjective onto B^n.

Since h and k are inverses, k is surjective onto J^n.

We still need to show that h and k are continuous. We will do this for h. The argument for k is similar. Let $\varepsilon > 0$ and $x \in J^n$ be given. We must show that there exists a $\delta > 0$ so that if $z \in B_\delta(x) \cap J^n$ then $h(z) \in B_\varepsilon(h(x)) \cap B^n$. If $x \neq 0$, the existence of such a δ follows immediately from the continuity of the formula used to define h. So suppose that $x = 0$. Let $\delta = \varepsilon/n$. Assume that $d(x,z) < \delta$. If $z = 0$, it is clear that $z \in B_\varepsilon(x)$. So suppose that $z \neq 0$. Then,

$$d(x,z) = d(0,z) = ||z|| < \delta.$$

Also,

$$d(h(x),h(z)) = d(0,h(z)) = ||h(z)|| = \frac{||z||_s}{||z||}||z|| = ||z||_s.$$

Now,

$$||z||_s = \max_i\{|z_i|\} \leq \sum |z_i| \leq \sum_i ||z|| = n||z|| < n\delta = \varepsilon.$$

Thus, If $d(x,z) < \delta$ then $d(h(x),h(z)) < \varepsilon$ implying that h is continuous.

A similar argument shows that k is continuous. Since h and k are inverses, B^n and J^n are homeomorphic. □