Exercise 1.4. Let $X = \{a, b, c, d\}$. Find all topologies on X containing five or fewer sets and prove that your list is complete.

Proof. (1) the indiscrete topology

(2) $\Big\{\varnothing, X, A\Big\}$ for any set $A \in \mathcal{P}(X) - \{\varnothing, X\}$. (14 possibilities)

(3) $\Big\{\varnothing, X, \{x, y\}, \{x\}\Big\}$ for $x, y \in X$ with $x \neq y$ (8 possibilities)

(4) $\Big\{\varnothing, X, \{x, y, z\}, \{x\}\Big\}$ for $x, y, z \in X$ no two identical. (9 possibilities)

(5) $\Big\{\varnothing, X, \{x, y, z\}, \{w\}\Big\}$ for $\{x, y, z, w\} = X$. (7 possibilities)

(6) $\Big\{\varnothing, X, \{x, y\}, \{z, w\}\Big\}$ for $\{x, y, z, w\} = X$. (6 possibilities)

(7) $\Big\{\varnothing, X, \{x, y, z\}, \{x, y\}\Big\}$ for $x, y, z \in X$ no two identical. (7 possibilities)

(8) $\Big\{\varnothing, X, \{x, y\}, \{x\}, \{y\},\Big\}$ for $x, y \in X$ with $x \neq y$. (6 possibilities)

(9) $\Big\{\varnothing, X, \{x, y, z\}, \{y, z, w\}, \{y, z\}\Big\}$ for $\{x, y, z, w\} = \{a, b, c, d\}$. (4 possibilities)

(10) $\Big\{\varnothing, X, \{x, y, z\}, \{x, y\}, \{z, w\}\Big\}$ for $\{x, y, z, w\} = \{a, b, c, d\}$ (10 possibilities)

(11) $\Big\{\varnothing, X, \{x, y, z\}, \{x, y\}, \{x\}\Big\}$ for $x, y, z \in X$ no two identical. (9 possibilities)

(12) $\Big\{\varnothing, X, \{x, y, z\}, \{y, z\}, \{x\}\Big\}$ for $x, y, z \in X$ no two identical. (9 possibilities)

(13) $\Big\{\varnothing, X, \{x, y, z\}, \{z, w\}, \{z\}\Big\}$ for $\{x, y, z, w\} = \{a, b, c, d\}$. (7 possibilities)

It is easy to see that all of these are topologies and that they are all distinct. It is clear that the only topologies containing two and three sets are the ones listed. Let $\mathcal{T}$ be a topology containing four sets. If it contains two singletons, it must contain their union implying it has five sets. Thus it contains at most one singleton; suppose that $\mathcal{T}$ contains a singleton. Let $A \in \mathcal{T}$ which is not $\varnothing, X$, or the singleton. Then A has either two or three elements. Suppose that A has two elements. There is no set with three elements and so the singleton must be an element of A. This is Option 3. Suppose that A has three elements. Either the singleton is in A or it isn't. These are options 4 and 5. Suppose that $\mathcal{T}$ has four sets and no singleton. Let A and B be the two sets which are not $\varnothing$ and X. If A and B each have two elements, then A and B must be disjoint since they are not equal and there is no singleton in $\mathcal{T}$. This is option 6. Suppose that A has three elements. If B has three elements then A and B are not disjoint and so $\mathcal{T}$ contains more than four sets. Thus, B must have two elements and must be a subset of A. This is option 7.

Suppose that $\mathcal{T}$ has five sets. By the arbitrary unions axiom, at most two of the sets can be singletons. If two of them are singletons then is the only option. Suppose that $A \in \mathcal{T}$ has three elements. If $B \neq A$ is in $\mathcal{T}$ and also has three elements then A and B have two elements in common. This two element set must be in $\mathcal{T}$. This is option 9. Suppose, therefore that $B \in \mathcal{T}$ and that B has two elements. Let C be the fifth set of $\mathcal{T}$. If C has two elements then either B and C are disjoint which is option 10 or they share a singleton in common. This singleton must be in $\mathcal{T}$ implying that $\mathcal{T}$ has six sets, a contradiction. Since we have already covered the situation when $\mathcal{T}$ has two three element sets, we may assume that C is a singleton. If B is a subset of A then this singleton may either be in B or in $A - B$. These are options 11 and 12. Suppose that B is not a subset of A. Then $C = A \cap B$ is a singleton. This is option 13. These are all the possibilities for topologies on X when $\mathcal{T}$ has five sets. $\square$