Topology Exam 1 Study Guide

(A.) Know precise definitions of the following terms

topology	topological space
metric	base for a topology
metric space	generating set for a topology
continuous	homeomorphism
subspace topology	product topology
equivalence relation	group
quotient topology	Hausdorff
connected	path connected
cover	compact
limit point of a sequence	limit point of a set
sequentially continuous	sequentially compact
Lebesgue Number of an open cover	order of a finite closed cover
affine subspace	affinely independent
topological dimension	n–dimensional simplex

(B.) Be able to answer questions about the following theorems. I will not ask you for complete proofs:

(a) The product of two compact spaces is compact. (What is an outline of the proof?)

(b) The Extreme Value Theorem. (What are the main ideas of the proof?)

(c) Jordan Curve Theorem. (What is the statement? What functions are useful in proving the theorem? How do you prove those functions are well-defined, continuous, etc.?)

(d) Sperner's Lemma. (What is the statement? What theorems are useful in proving it?)

(e) Invariance of Dimension. (What is the statement? What is an outline of the proof? What are the main ideas used in each step?)

(C.) Be able to give correct proofs of the following result. I will not ask you to prove anything not on this list.

(a) Suppose that X, Y and Z are topological spaces and that $h\colon Y \to Z$, $f\colon X \to Y$, $g\colon X \to Z$ are functions with $g = h \circ f$. Suppose that g is continuous and that f is open. Prove that h is continuous.

(b) "Crushing" is an equivalence relation.

(c) A topological space X has at least n connected components if and only if there exists a continuous surjective function of X onto a discrete space with n points.

(d) If X is path-connected then it is connected.
(e) If X is a closed and bounded subset of $\mathbb{R}^n$ then X is compact. You may use the fact that $[0,1]$ is compact and that the product of compact spaces is compact.
(f) Suppose that X is a metric space and that $A \subset X$. Prove that $\overline{A}$ is the smallest closed set in X containing A.
(g) Let $M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries. Give $M_n(\mathbb{R})$ the usual topology on $\mathbb{R}^{n^2}$. You may use the fact that the determinant function (being a polynomial) is continuous.
   (i) $GL_n(\mathbb{R})$, the set of invertible $n \times n$ matrices, is an open subset of $M_n(\mathbb{R})$.
   (ii) (Bonus:) Let A be an invertible $n \times n$ matrix. There exists $\varepsilon > 0$ so that if B is an $n \times n$ matrix such that if, for all $1 \leq i, j \leq n$,
$$|B_{ij} - A_{ij}| < \varepsilon$$
   then B is invertible.
   (iii) $SL_n(\mathbb{R})$, the set of $n \times n$ matrices with determinant 1 is a closed subset of $M_n(\mathbb{R})$.
   (iv) $GL_n(\mathbb{R})$ is not connected.
(h) Suppose that X is a compact metric space and that $A \subset X$ is compact. Then the topological dimension of A is no more than the topological dimension of X.
(i) Let $\sim$ be an equivalence relation on X. Let $f\colon X \to X/\!\sim$ be the quotient map. Then f is continuous and open.
(j) Suppose that X is compact and that $\sim$ is an equivalence relation on X. Then $X/\!\sim$ is compact.
(k) Suppose that X is a topological space with infinitely many connected components. Then X is not compact.