Lecture Notes from February 25, 2009

Lemma 1. Let $(X,\mathbb{X})$, $(Y,\mathbb{Y})$, and $(Z,\mathbb{Z})$ be topological spaces and suppose that

$$f\colon X\to Y \qquad \text{and} \qquad g\colon X\to Z$$

are continuous. Also suppose that f is an open function and that $h\colon Y\to Z$ is a function such that

$$g=h\circ f$$

Then h is continuous.

Recall that a function is open if the image of every open set is open.

Proof. Let $U\subset Z$ be open. We need to show that $h^{-1}(U)$ is open. Since g is continuous, $g^{-1}(U)$ is open. Since f is an open function, $f(g^{-1})(U)$ is open. Since $g=h\circ f$, $h^{-1}(U)=f(g^{-1}(U))$. Thus, $h^{-1}(U)$ is open. $\square$

Lemma 2. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $f\colon X\to X/{\sim}$ be the quotient function defined by

$$f(x)=[x].$$

If $X/{\sim}$ is given the quotient topology, then f is an open function.

Proof. Let $U\subset X$ be an open set. The quotient topology on $X/{\sim}$ consists of all sets $V\subset X/{\sim}$ such that $f^{-1}(V)$ is open. Since $f^{-1}(f(U))=U$, the set $f(U)$ is open. Hence, f is an open function. $\square$

Lemma 3. For each $i\in\{1,\ldots,n\}$ let X_i and Y_i be topological spaces and let $f_i\colon X_i\to Y_i$ be a continuous function. Then the function

$$f=\times_{i=1}^n f_i\colon\ \times_{i=1}^n X_i\to\times_{i=1}^n Y_i$$

defined by

$$f\big((x_1,\ldots,x_n)\big)=(f_1(x_1),\ldots,f_n(x_n))$$

is continuous.

Proof. Sets of the form $\times_{i=1}^n V_i$ with $V_i\subset Y_i$ open form a base for the product topology on $\times_{i=1}^n Y_i$. It, therefore, suffices to check that for each such set

$$f^{-1}(\times_{i=1}^n V_i)$$

is open. Notice that

$$f^{-1}(\times_{i=1}^n V_i)=\times_{i=1}^n f_i^{-1}(V_i).$$

Since each f_i is continuous, by the definition of the product topology $f^{-1}(\times_{i=1}^n V_i)$ is open. $\square$

Corollary 4. Let X, Y_1, and Y_2 be topological spaces and let $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$ be continuous functions. Then $f\colon X \to Y_1 \times Y_2$ defined by

$$f(x) = (f_1(x), f_2(x))$$

is continuous.

Proof. Define $i\colon X \to X \times X$ by $i(x) = (x,x)$. Notice that i is continuous. The function f is equal to $(f_1 \times f_2) \circ i$. Since it is the composition of two continuous functions, it is continuous. □

Theorem 5. The topological space $[0,2\pi]/\{0,2\pi\}$ is homeomorphic to S^1.

Proof. Define $g\colon [0,2\pi] \to S^1 \subset \mathbb{R}^2$ by

$$g(x) = ((\cos x, \sin x)).$$

By Corollary 4, g is continuous. Let $S = [0,2\pi]/\{0,2\pi\}$. And let $f\colon [0,2\pi] \to S$ be the quotient map. By Lemma 2 f is continuous and open. Define $h\colon S \to S^1$ by

$$h([x]) = (\cos x, \sin x).$$

If $[x] \in S$ then either $[x] = \{x\}$ or $[x] = \{0,2\pi\}$. Since $(\cos 0, \sin 0) = (\cos 2\pi, \sin 2\pi)$ the function h is well-defined. Also, since

$$h^{-1}\big((\cos x, \sin x)\big) = \{x\} \text{ or } \{0,2\pi\} = [0]$$

the function h is injective. It is also easy to see it is surjective.

The function h is continuous by Lemma 1. It remains to show that h^{-1} is continuous.

Notice that if an open set $U \subset [0,2\pi]$ contains both 0 and 2π, then $g(U) \subset S^1$ is open. Also note that if $U \subset [0,2\pi]$ is an open set which contains neither 0 nor 2π, then $g(U)$ is open. To show that h^{-1} is continuous, we will mimic the proof of Lemma 1. Let $U \subset S$ be an open set. We need to show that $h^{-1}(U)$ is open. Notice that $h^{-1}(U) = g(f^{-1}(U))$. Since f is continuous, $f^{-1}(U)$ is an open set in $[0,2\pi]$. Furthermore, $f^{-1}(U)$ is an open set with contains either both 0 and 2π or neither 0 and 2π. Thus, $g(f^{-1}(U))$ is open in S^1. Hence, h^{-1} is continuous. □

Corollary 6. Let $S = [0,2\pi]/\{0,2\pi\}$. Then $S \times S$ is homeomorphic to $T^2 = S^1 \times S^1$.

Proof. Let $h\colon S \to S^1$ be a homeomorphism. Then $h \times h\colon S \times S \to T^2$ will also be a homeomorphism. □