Theorems on Compactness

These statements and proofs are based on theorems from Bredon's *Topology and Geometry*.

We begin by proving that the space $[0,1]$ is compact.

Theorem 1. With the subspace topology, $[0,1] \subset \mathbb{R}$ is compact

Proof. Let $\mathscr{U}$ be an open cover of $[0,1]$. Define

$$S = \{s \in [0,1] : \text{ there is a finite subset of } \mathscr{U} \text{ which covers } [0,s]\}.$$

Let b be the least upper bound for S.

Claim : $S = [0,b)$ or $S = [0,b]$
To see this, suppose that $s \in S$. Let $\mathscr{U}'$ be a finite subset of $\mathscr{U}$ which covers $[0,s]$. Then for all $s' < s$, $\mathscr{U}'$ also covers $[0,s']$. Hence, if $s \in S$, then $[0,s] \subset S$. □

Claim: $S = [0,b]$.
Suppose not; that is, suppose that $S = [0,b)$. Since $\mathscr{U}$ is an open cover of $[0,1]$, there exists an open set $U \in \mathscr{U}$ so that $b \in \mathscr{U}$. Since U is an open set in $[0,1]$, there exists $\varepsilon > 0$ so that the interval $(b-\varepsilon, b]$ is a subset of U. Since b is the least upper bound for S, the number $b - \varepsilon/2$ is contained in S. Let $\mathscr{U}'$ be a finite subset of $\mathscr{U}$ which covers $[0, b-\varepsilon/2]$. Then $\mathscr{U}' \cup \{U\}$ is a finite subset of U' which covers $[0,b]$. Hence, $b \in S$ and so, $S = [0,b]$. □

Claim: $S = [0,1]$. Suppose not. That is, suppose that $b < 1$. Let $\mathscr{U}'$ be a finite subset of $\mathscr{U}$ which covers $S = [0,b]$. Since each set of $\mathscr{U}$ is open in $[0,1]$, there exists a set $U \in \mathscr{U}$ so that $b \in U$. Since U is open and since $b < 1$, there exists $\varepsilon > 0$ so that $(b, b+\varepsilon) \subset U$. Thus, $\mathscr{U}'$ is a finite subset of $\mathscr{U}$ which covers $[0, b+\varepsilon/2]$. This implies that $b + \varepsilon/2 \in S$. But, b is the least upperbound for S and so this is impossible. □

Thus, $[0,1]$ is compact. □

Lemma 2. If X is compact and if $A \subset X$ is closed, then A is compact.

Proof. Let $\mathscr{U}$ be an open cover of A. Since $A \subset X$ is compact, $X - A$ is open. Thus, $\mathscr{U} \cup \{X - A\}$ is an open cover of X. Since X is compact, there exists a finite subcover. Let $\mathscr{U}'$ be that finite subcover minus the set $X - A$, if $X - A$ was in that subcover. Then $\mathscr{U}'$ is a finite subcover of $\mathscr{U}$ which covers A. Hence, A is compact. □

We now proceed to show that the product of two compact spaces is compact.

Definition 3. If Z and Y are compact topological spaces and if $f\colon Z \to Y$ is a function then f is **proper**, if for each compact set $C \subset Y$, $f^{-1}(C)$ is compact. The function f is **open**, if for each open set $U \subset Z$, the set $f(U)$ is open in Y.

Lemma 4. Suppose that Z and Y are topological spaces and that $f\colon Z \to Y$ is an open function. Suppose that for each $y \in Y$, $f^{-1}(y)$ is compact in X. Then f is a proper function.

Proof. Let $C \subset Y$ be compact. We wish to show that $f^{-1}(C)$ is compact in Z. Let $\{U_\alpha : \alpha \in A\}$ be an open cover, with A an index set. For each $y \in C$, $f^{-1}(y)$ is compact. Let $A_y \subset A$ be a finite subset so that $\{U_\alpha : \alpha \in A_y\}$ is a finite cover of $f^{-1}(y)$. Since each U_α is open, the set

$$W_y = \bigcup_{\alpha \in A_y} U_\alpha$$

is open in Z.

Since f is an open function, the image of a closed set is closed. Hence, $f(Z - W_y)$ is a closed set in Y. Thus,

$$V_y = Y - f(Z - W_y)$$

is an open set in Y.

Claim: For each $y \in C$, $y \in V_y$. Suppose that y is not in V_y. Then $y \in f(Z - W_y)$. This means that there exists $z \in f^{-1}(y)$ such that $z \in Z - W_y$. However, there exists $\alpha \in A_y$ so that $z \in U_\alpha$. Furthermore, this U_α is a subset of W_y. So $z \in U_\alpha \subset W_y$. This means that z is not in $Z - W_y$, a contradiction.

Hence, $\{V_y\}$ is an open cover of the compact set C. Thus, there exist points $y_1, \ldots, y_n \in C$ so that $\{V_{y_i} : 1 \le i \le n\}$ is a finite open cover of C.

Claim: For each y_i, $f^{-1}(V_{y_i}) \subset W_{y_i}$. Suppose that $z \in f^{-1}(V_{y_i})$. Then, $f(z) \in Y - f(Z - W_y)$. Hence z is not in $Z - W_y$ which means that $z \in W_y$. □

Since $C \subset \bigcup V_{y_i}$, we have

$$f^{-1}(C) \subset \bigcup_{i=1}^{n} f^{-1}(V_{y_i}) \subset \bigcup_{i=1}^{n} W_{y_i} = \bigcup_{i=1}^{n} \bigcup_{\alpha \in A_{y_i}} U_\alpha.$$

Thus,

$$\{U_\alpha : \text{ there exists } 1 \le i \le n \text{ so that} \alpha \in A_{y_i}\}.$$

This is a finite subset of $\{U_\alpha : \alpha \in A\}$ and it covers $f^{-1}(C)$, so $f^{-1}(C)$ is compact and f is proper. □

Lemma 5. Let X and Y be topological spaces. Then the projection $\pi_Y: X \times Y \to Y$ is an open function.

Proof. Let $W \subset X \times Y$ be an open set. Since products of open sets from X and Y form a basis of the product topology, there exist open sets $U_\alpha \subset X$ and $V_\alpha \subset Y$ for α in some index set A so that $W = \bigcup U_\alpha \times V_\alpha$. Then,

$$\pi_Y(W) = \pi_Y(\bigcup U_\alpha \times V_\alpha) = \bigcup \pi_Y(U_\alpha \times V_\alpha) = \bigcup V_\alpha$$

This last set is open in Y. □

Lemma 6. Suppose that X and Y are topological spaces and that X is compact. Then for each $y \in Y$, $\pi_Y^{-1}(y) \subset X \times Y$ is compact.

Proof. For each $y \in Y$,

$$\pi_Y^{-1}(y) = X \times \{y\}$$

This is homeomorphic to X and so is compact. □

Theorem 7. Suppose that X and Y are compact topological spaces. Then $X \times Y$ is compact.

Proof. The projection $\pi_Y: X \times Y \to Y$ is open and the inverse image of a point in Y is compact. Thus, by Lemma 4, π_Y is proper. Since Y is compact, $\pi_Y^{-1}(Y) = X \times Y$ is compact. □

Corollary 8. If $C \subset \mathbb{R}^n$ is a closed and bounded set, C is compact.

Proof. Since C is bounded, there exists an interval $[a,b] \in \mathbb{R}$ so that

$$C \subset \times_{i=1}^n [a,b] \subset \mathbb{R}^n.$$

Since the product of compact spaces is compact $\times_{i=1}^n [a,b]$ is compact. Hence, C is a closed subset of a compact set. By Lemma 2, C is compact. □

Theorem 9 (Extreme Value Theorem)**.** Let X be a compact topological space and suppose that $f: X \to \mathbb{R}$ is a continuous function. Then there exists $M \in X$ so that for all $x \in X$, $f(x) \leq f(M)$. Similarly, there exists $m \in X$ so that for all $x \in X$, $f(m) \leq f(x)$.

Proof. We will only prove that M exists. Define M to be the least upper-bound of the set $f(X) \subset \mathbb{R}$. (Why does M exist?) If $M \in f(X)$, we are done, so suppose that $M \notin f(X)$. Define

$$U_n = \{(-\infty, M - \frac{1}{n}) \subset \mathbb{R}\}$$

Clearly, each U_n is open in $\mathbb{R}$. If $x \in X$, then by definition of the least upperbound, $f(x) \leq M$. By the assumption that $M \notin f(X)$, $f(x) < M$. Thus there exists $k \in \mathbb{N}$, such that $f(x) < M - \frac{1}{k}$. Consequently, $f(x) \in U_k$ and so $\{U_n\}$ is an open cover of $f(X)$. Since X is compact, $f(X)$ is compact. Hence, there is a finite subset of $\{U_n\}$ which covers $f(X)$. Since $U_n \subset U_{n+1}$ for all $n \in \mathbb{N}$, this means that there exists $N \in \mathbb{N}$ so that $f(X) \subset U_N$. In particular,

$$f(x) < M - \frac{1}{N} \qquad \text{for all } x \in X.$$

Thus, $M - \frac{1}{N}$ is an upper bound for $f(X)$ which is strictly smaller than M. But this contradicts the fact that M is the least upper bound for $f(X)$. Thus, $M \in f(X)$ and f achieves its maximum. □

Definition 10. Suppose that (X,d) is a metric space and that A and B are non-empty subsets. Define

$$d(A,B) = \inf\{d(a,b) : a \in A \text{ and } b \in B\}.$$

Since d is a metric, this is a well defined function with values in $[0,\infty) \subset \mathbb{R}$.

We can use the extreme value theorem to prove:

Lemma 11. Suppose that (X,d) is a metric space and that $A,B \subset X$ are non-empty subsets with B compact. Then there exists a point $b_0 \in B$ such that $d(A,B) = d(A,b_0)$.

Proof. Fix $a \in A$. The function $d(a,\cdot)\colon B \to \mathbb{R}$ is a continuous real valued function on a compact set. It therefore achieves its minimum (by the extreme value theorem). Let $b_0 \in B$ be the point such that $d(a,b_0)$ is the minimum of $d(a,\cdot)$. Then, for all $a \in A$, $b \in B$:

$$d(b_0,a) \leq d(b,a)$$

which implies that for all $b \in B$:

$$d(b_0,A) = \inf\{d(b_0,a) : a \in A\} \leq \inf\{d(b,a) : a \in A\}.$$

This in turn implies

$$d(b_0,A) \leq \inf_{b \in B} \inf_{a \in A}\{d(b,a)\} = \inf\{d(b,a) : b \in B, a \in A\} = d(A,B).$$

. □

Lemma 12. If (X,d) is a compact metric space and if $x \in X$ and if B is a closed subset of X with $x \notin B$ then $d(x,B) > 0$.

Proof. Suppose not. Since B is closed, $X \backslash B$ is open. Hence, for each $x \in X \backslash B$, there is a ball $B_{\delta(x)}(x)$ with $\delta > 0$ such that $B_\delta(x) \subset X \backslash B$. Then, for all $y \in B$,

$$d(x,y) > \delta(x).$$

Hence, $d(x,B) \geq \delta(x) > 0$. □

Definition 13. Suppose that V is a subset of a metric space (X,d). The **diameter** of V is

$$\operatorname{diam}(V) = \sup\{d(x,y) : x,y \in V\}$$

Theorem 14 (Lebesgue Covering Lemma)**.** Let (X,d) be a compact metric space and suppose that $\{U_\alpha : \alpha \in A\}$ is an open cover of X. Then there exists $\delta > 0$ such that if $V \subset X$ and $\operatorname{diam}(V) < \delta$ then there exists $\alpha \in A$ so that $V \subset U_\alpha$.

Proof. If $x \in X$ then there exists α so that $x \in U_\alpha$. Choose $\varepsilon(x) > 0$ so that $B_{2\varepsilon(x)}(x) \subset U_\alpha$. (Notice that $\varepsilon(x)$ also depends on α.). The set $\mathscr{B} = \{B_{\varepsilon(x)}(x) : x \in X\}$ is an open cover of X. Since X is compact, there exists a finite subset of $\mathscr{B}$ which also covers X. That is, there exist points $x_1, x_2, \ldots, x_n$ so that $\{B_{\varepsilon(x_i)} : 1 \leq i \leq n\}$ is an open cover of X. Define

$$\delta = \min\{\varepsilon(x_i) : 1 \leq i \leq n\}.$$

Suppose that $V \subset X$ and that $\operatorname{diam}(V) < \delta$. We wish to show that there exists α such that $V \subset U_\alpha$. Let $x \in V$. There exists i so that $x \in B_{\varepsilon(x_i)}(x_i)$. That is, $d(x,x_i) < 2\varepsilon(x_i)$. Let $y \in V$. We desire to show that $y \in B_{2\varepsilon(x_i)}(x_i)$. We know that

$$d(y,x_i) \leq d(y,x) + d(x,x_i) \leq \delta + \varepsilon(x_i) \leq 2\varepsilon(x_i).$$

Thus, $y \in B_{2\varepsilon(x_i)}(x_i)$. Hence, $V \subset B_{2\varepsilon(x_i)}(x_i) \subset U_\alpha$ for some α. □

For a fixed open cover $\{U_\alpha\}$, the least upperbound of the numbers δ in the statement of the previous lemma is called the **Lebesgue number** of the cover.

Lemma 15. Suppose that (X,d) is a compact non-empty metric space. Then the Lebesgue number of any open cover of X is non-zero.

Proof. Let $\mathscr{U}$ be an open cover of X. Since X is compact, there is a finite subcover $\mathscr{U}' \subset \mathscr{U}$. Let $\mathscr{U}' = \{U_1, \ldots, U_n\}$. For $x \in X$ define

$$f_i(x) = d(x, X \backslash U_i).$$

Recall that if $x \in U_i$ then $f_i(x) > 0$. Define

$$f(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$

The function f is continuous. Also, if $x \in X$, then $f(x) > 0$. To see this, notice that since $\mathscr{U}'$ is a cover of X, for each $x \in X$, there exists i so that $x \in U_i$. Then $f_i(x) > 0$.

Thus, $f(X) \subset \mathbb{R}$ does not contain 0. Since X is compact, by the extreme value theorem, there exists $m \in X$ so that $f(m) = \inf f(X)$. In particular, $0 \neq \inf f(X)$. Hence, there exists $\varepsilon > 0$ so that for each $x \in X$, $f(x) \geq f(m) > \mu > 0$.

Suppose that $V \subset X$ is a subset with $\operatorname{diam}(V) < \mu$. Let $x \in V$. There exists i so that $f_i(x) > \mu$. That is, $d(x, X \backslash U_i) > \mu$. In other words, $B_\mu(x) \subset U_i$. Since $\operatorname{diam}(V) < \mu$,

$$V \subset B_\mu(x) \subset U_i.$$

Hence, the Lebesgue number for $\mathscr{U}$ is at least $\mu > 0$. □