We have seen various ways of describing the groups D_n and $\mathbb{S}_n$ using different generating sets. In general, it is a difficult problem to explicitly describe a given group, whether or not it is a group of symmetries. Using an extension of LaGrange's theorem, though, we can learn somethings.

Definition 4. Suppose that G is a group of symmetries of an object X. Let x be a point of X. Define the orbit of x to be the set of all points y in X. such that there is a symmetry in G which takes x to y. Denote this set by $\mathtt{orb}_G(x)$.

Let x be a vertex of the square. The group D_4 is the group of all symmetries of the square. A symmetry in D_4 takes x to another vertex, and we can send x to any vertex we want to by choosing an appropriate symmetry in D_4. Thus, $\mathtt{orb}_{D_4}(x)$ is the set of vertices of the square.

The group $G = \{\mathbf{I}, R_{180}\}$ is a subgroup of D_8. The set $\mathtt{orb}_G(x)$ consists of the vertex x and the vertex directly opposite it on the square.

Definition 5. Suppose that G is a group of symmetries of an object X. Let x be a point of X. The set of group elements g which don't move x is called the **stabilizer** of x in G. It is denoted $\mathtt{stab}_G(x)$.

Let x be the upper left vertex of the square, then $\mathtt{stab}_{D_4}(x) = \{\mathbf{I}, D\}$ since every element of D_4 except the identity and the diagonal reflection moves x to some other vertex. If x is the center of the square, then $\mathtt{stab}_{D_4}(x) = D_4$, since no element of D_4 moves the center of the square.

Exercise 16. Prove that $\mathtt{stab}_G(x)$ is a subgroup of G for any given point x.

Theorem 5. (Orbit-Stabilizer) Suppose that G is a group of symmetries of an object X. For any point x in X,

$$|\mathtt{orb}_G(x)| \cdot |\mathtt{stab}_G(x)| = |G|.$$

Proof. We simply need to show that $|\mathtt{orb}_G(x)| = [G : \mathtt{stab}_G(x)]$. Let $[g]$ be a coset of $\mathtt{stab}_G(x)$ in G. Match the coset $[g]$ with the point $g(x)$ in $\mathtt{orb}_G(x)$. Notice that if g and g' are both in $[g]$ then we have $g' = g \circ h$ for some h in $\mathtt{stab}_G(x)$. Then $g'(x) = g \circ h(x)$. Since $h(x) = x$, $g'(x) = g(x)$ and so this matching is well defined. Notice also that ever point in the orbit of x is matched with some coset and that if $g(x) = g'(x)$ then $g^{-1} \circ g'(x) = x$. This implies that $g^{-1} \circ g$ is in $\mathtt{stab}_G(x)$. It turns out that this implies that $[g] = [g']$.

Thus, each coset of $\mathtt{stab}_G(x)$ is matched with one point in $\mathtt{orb}_G(x)$ and different cosets are matched with different points. Since each point of the orbit of x is matched with some coset, the size of $\texttt{orb}_G(x)$ is equal to the number of cosets which is $[G : \texttt{stab}_G(x)]$. $\square$

Let's use this to do something interesting.

Let X be a cube in 3–space and let $G = \texttt{Sym}(X)$. Let's determine how many symmetries are in G. Let x be a vertex of the cube. There is a symmetry which sends x to any other vertex that we want. Every symmetry of X must send vertices to vertices. The cube has 8 vertices, so $|\texttt{orb}_G(x)| = 8$. Suppose that T is a symmetry of the square which doesn't move x. There are three edges coming into x and T must permute those in some way. It is not hard to see that all possibilities for permutations can be achieved. Furthermore, if T fixes x and all edges coming into x then T must be $\mathbf{I}$. To see this, recall that if a symmetry of 3–space fixes three points then it is the identity.

Thus,

$$|\texttt{stab}_G(x)| = |\mathbb{S}_3| = 6.$$

Hence, by the orbit-stabilizer theorem, G has $8 \cdot 6 = 48$ elements.

The five platonic solids are the tetrahedron, the cube, the octahedron, the dodecahedron, and the iscosahedron.

Exercise 17. Look up pictures of each of the platonic solids and perform an analysis similar to what we just did to determine the orders of their groups of symmetries.

Some of the symmetries that we counted include reflections. Let $\texttt{Sym}^+(X)$ denote the subgroup of $\texttt{Sym}^+(X)$ which preserve the orientation of 3–space. Let $[g]$ and $[h]$ be cosets of this group in $\texttt{Sym}(X)$ where both g and h reverse orientation. Notice that g^{-1} reverses orientation. Notice also that, $k = g^{-1} \circ h$ must preserve orientation; that is, $g \circ h$ is in $\texttt{Sym}^+(X)$. The symmetry k^{-1} also preserves orientation. Thus,

$$h \circ k^{-1} \text{ is in } [h].$$

But

$$h \circ k^{-1} = h \circ (g^{-1} \circ h)^{-1} = h \circ h^{-1} \circ g = g.$$

Hence, $[g] = [h]$. This proves that there are at most two cosets of $\texttt{Sym}^+(X)$ in $\texttt{Sym}(X)$: $\texttt{Sym}^+(X)$ and one other one $[g]$. Thus, the index of $\texttt{Sym}^+(X)$ is either one or two in $\texttt{Sym}^+(X)$. We can conclude, for example, that there are 24 orientation preserving symmetries of the cube. Figure 4 depicts three representative examples.

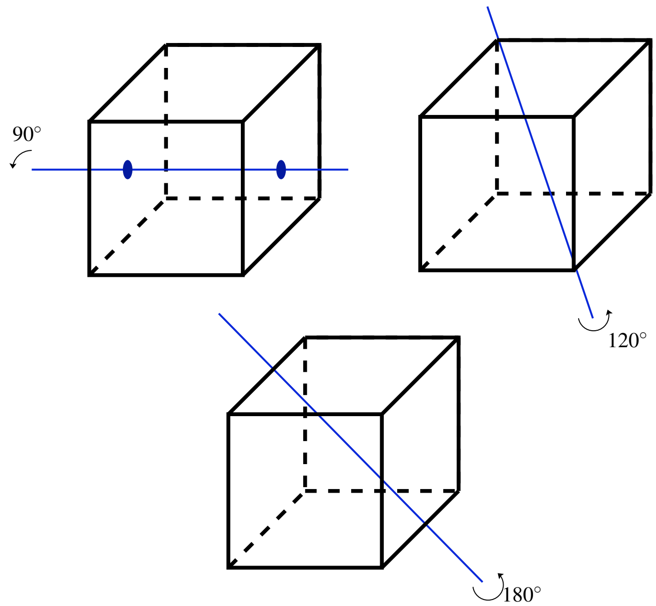

FIGURE 4. Examples of the three types of orientation preserving symmetries of the cube.