Lecture Notes on Symmetry

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1. Preliminaries

A **mapping** or **transformation** of a set X is a function which takes each point of X to some other point of X. A transformation is a **automorphism** or **symmetry** of X if it preserves a given "structure" of X. For example, automorphisms of the xy-plane, preserve distance: the distance between points A and B before the automorphism is the same as the distance between A' and B' after the automorphism (the automorphism takes A to A' and B to B'.) We always require our transformations and automorphisms to be "one-to-one". This means that two distinct points A and B are never mapped to the same point $A' = B'$.

Two important examples of situations where we care about automorphisms are:

(a) Symmetries of the plane (which preserve distance)
(b) Symmetries of a graph which preserve the relationship between vertices and edges.

We'll discuss symmetries of graphs more later, but we'll first study symmetries of the plane. It turns out that every symmetry of the plane is either a reflection (about some line), a rotation (about some point of with some angle), a translation (in some direction by some distance), or a combination of these.

Our first important example will concern symmetries of a square in the plane. This means we want to list all symmetries of the plane which take the square to itself. In other words, points on the square can be moved about within the square, but not outside the square. Two symmetries are considered to be the same if they have the same effect on the points of the square. For example a 90° rotation clockwise is the same as a 270° rotation counterclockwise, because all the points end up in the same spot. This will make more sense when we look at specific examples.

2. GROUPS

2.1. **Group Tables.** Consider the square below. On the left is the plain old square; on the right some axes of reflection are drawn. Reflecting the square about one of these axes produces a square which is indistinguishable from the first. We call the act of reflecting the square across one of these lines, a **reflection symmetry**.

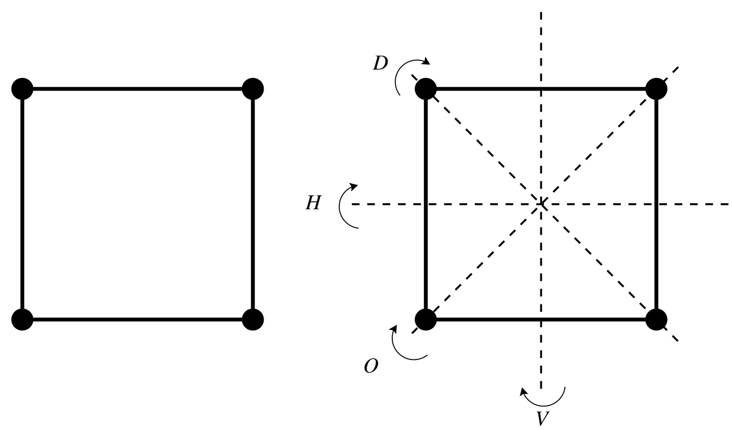

FIGURE 1. The symmetries of the square

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In addition to the reflection symmetries, we can also rotate the square by multiples of 90° either clockwise or counter-clockwise. Denote a counterclockwise rotation of θ degrees by R_θ. Figure 2 shows the effects of repeatedly applying R_{90}. For example, performing R_{90} once moves the purple vertex from the upper right to the upper left and cycles the other colored vertices around ”one notch”.

At this point, we should be a little more precise. A **symmetry** of an object is a way of moving the object so that after the motion the object cannot be distinguished from the the way it was before the motion. When we discuss shapes (like a square) lying on the plane we will insist that the motion not change the distance between two arbitrary points. Sometimes, for other objects, we will not insist that distance remain unchanged. It will usually be clear from the context whether or not we assume distances are unchanged.

Even though a symmetry is a motion or action, we will usually think of it as an object of study in its own right. To be able to tell two different symmetries apart we will often decorate the object (e.g. the square) and

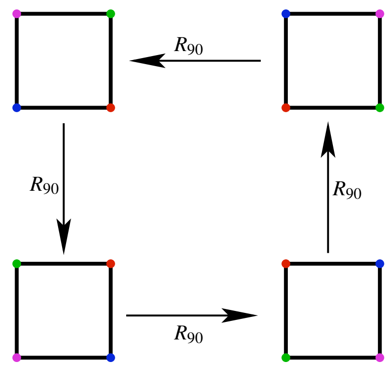

FIGURE 2. Applying R_{90} to the square. The vertices have been colored to exhibit the effect of R_{90}.

look at what happens to the decorations. For example, the rotation R_{90} moves the colors of the vertices counter-clockwise. Two symmetries are ”the same” if they have the same effect on our decorations. For example, performing R_{90} and then performing R_{180} is the same as performing R_{270}. Similarly, performing R_{270} is the same as rotating the square by 90° in a *clockwise* direction.

So far, we have listed 7 symmetries of the square:

$$R_{90}, R_{180}, R_{270}, D, V, O, H.$$

In theory, we could produce new symmetries of the square by performing one of these symmetries. For example, performing R_{90} and then performing R_{90} again is the same as performing R_{180}. There is also the symmetry $\mathbf{I}$, which consists of doing nothing at all. If S_1 and S_2 are symmetries, if we first perform S_1 and then perform S_2 we call the resulting symmetry $S_2 \circ S_1$. Notice that we should read this expression right to left.

Question: Is our list of symmetries: $\mathbf{I}$, R_{90}, R_{180}, R_{270}, D, V, O, H complete?

Recall that D, V, O, and H are the reflections of the square about the lines indicated in Figure 1. Let’s make a table:

S_1

	$S_2 \circ S_1$	$\mathbf{I}$	R_{90}	R_{180}	R_{270}	D	V	O	H
	$\mathbf{I}$								
	R_{90}								
	R_{180}								
S_2	R_{270}								
	D								
	V								
	O								
	H								

To fill in the table, notice that we must have $S \circ \mathbf{I} = S$ no matter what symmetry S is, since $\mathbf{I}$ means do nothing. Similarly, $\mathbf{I} \circ S = S$ no matter what symmetry S is. For example, $\mathbf{I} \circ R_{90} = R_{90}$ since rotating by $90°$ and then doing nothing is the same as rotating by $90°$. This allows us to fill in the first row and the first column:

S_1

	$S_2 \circ S_1$	$\mathbf{I}$	R_{90}	R_{180}	R_{270}	D	V	O	H
	$\mathbf{I}$	$\mathbf{I}$	R_{90}	R_{180}	R_{270}	D	V	O	H
	R_{90}	R_{90}							
	R_{180}	R_{180}							
S_2	R_{270}	R_{270}							
	D	D							
	V	V							
	O	O							
	H	H							

Next, notice that if we perform R_{90} and then perform R_{90} we have simply rotated the square $180°$. That is, we have performed R_{180}. Using similar lines of reasoning we can fill in the upper left quadrant of the table:

S_1

	$S_2 \circ S_1$	$\mathbf{I}$	R_{90}	R_{180}	R_{270}	D	V	O	H
	$\mathbf{I}$	$\mathbf{I}$	R_{90}	R_{180}	R_{270}	D	V	O	H
	R_{90}	R_{90}	R_{180}	R_{270}	$\mathbf{I}$				
	R_{180}	R_{180}	R_{270}	$\mathbf{I}$	R_{90}				
S_2	R_{270}	R_{270}	$\mathbf{I}$	R_{90}	R_{180}				
	D	D							
	V	V							
	O	O							
	H	H							