square then a rotation will never change the fact that the arrows point counterclockwise. A reflection, however, does change the arrows from being counterclockwise to being clockwise. Thus, no combination of rotations can ever produce a reflection.

Exercise 7. Show that it is possible to generate D_n using only a reflection and a rotation. How many degrees must the rotation rotate? Does it matter what the reflection does?

Let's study the symmetric groups.

Exercise 8. What is the fewest number of elements of $\mathbb{S}_3$ that will generate $\mathbb{S}_3$? List several possibilities for generating sets with the fewest possible number of elements.

A **transposition** in a symmetric group $\mathbb{S}_n$ is a symmetry that swaps the position of two dots.

Theorem 2. The collection of all transpositions generates $\mathbb{S}_n$ for $n \geq 2$.

Proof. We must show that every permutation of n dots can be written as the combination of transpositions. This is clearly true for $n = 2$ and can easily be verified for $n = 3$ using Table 2.

Let T be in $\mathbb{S}_4$. Number the dots 1, 2, 3, 4. The effect of T on the dots can be written in the following form

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ T(1) & T(2) & T(3) & T(4) \end{array}$$

$T(1)$ is one of the numbers 1, 2, 3, or 4. Suppose first that $T(1) = 1$. Then T is a symmetry of the three dots labelled 2, 3, and 4 and therefore lives in $\mathbb{S}_3$. We have already seen that every symmetry in $\mathbb{S}_3$ can be written as a combination of transpositions. Thus, T can be written as a combination of transpositions.

Suppose that $T(1) \neq 1$. Let C be the 2–cycle $[1 \leftrightarrow T(1)]$. Let $S = C \circ T$. Then S can be described as:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ T(1) & T(2) & T(3) & T(4) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & C(T(2)) & C(T(3)) & C(T(4)) \end{array}$$

Notice, therefore that S is a symmetry of dots 2, 3, and 4. It is, therefore a product of transpositions. Notice that $C \circ C = \mathbf{I}$. We have

$$S = C \circ T$$

Thus,

$$\begin{aligned} C \circ S &= (C \circ C) \circ T \\ C \circ S &= \mathbf{I} \circ T \\ C \circ S &= T. \end{aligned}$$

Thus, T is the combination of transpositions. A similar argument shows that every symmetry in $\mathbf{S}_5$ is the combination of transpositions. We then boot strap our way to conclude that every symmetry in $\mathbb{S}_n$ is a combination of transpositions for any $n \geq 2$. $\square$

Exercise 9. How many transpositions are there in $\mathbb{S}_n$?

Exercise 10. Show that the following set of transpositions generate $\mathbb{S}_n$ for $n \geq 2$:

$$\begin{gathered} [1 \leftrightarrow 2] \\ [2 \leftrightarrow 3] \\ [3 \leftrightarrow 4] \\ \vdots \\ [n-1 \leftrightarrow n]. \end{gathered}$$

These are called **adjacent transpositions**.

Exercise 11. How many adjacent transpositions are there in $\mathbb{S}_n$?