If a group has only finitely many elements, in principle we can make up a group table like we did for the group of symmetries of a square. Here is another example. In this example the object will be three indistinguishable dots: $\bullet\bullet\bullet$. You should think of these points, which means that reflection about a horizontal line will not count as one of our symmetries. Our group will be the group of symmetries of these dots. We won't insist that the symmetry preserve the distance between the dots. One example of such a symmetry is swapping the first two dots. As with the square, we'll add some colors to the dots: $\textcolor{red}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}$. This will enable us to keep track of the different behaviour of different symmetries.

To recap: our group is $\mathbb{G} = \mathtt{Sym}(\bullet\bullet\bullet)$. We need names for the different symmetries of the dots. Denote the action of swapping the first two dots by $[1 \leftrightarrow 2]$. Notice that this changes the colors of the dots:

$$[1 \leftrightarrow 2](\textcolor{red}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}) = \textcolor{green}{\bullet}\textcolor{red}{\bullet}\textcolor{blue}{\bullet}.$$

We can also move the first dot to the second position, the second dot to the third position, and the third dot to the first position. Denote that symmetry as $[1 \to 2 \to 3 \to]$. Notice that this one also changes the colors:

$$[1 \to 2 \to 3 \to](\textcolor{red}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}) = \textcolor{blue}{\bullet}\textcolor{red}{\bullet}\textcolor{green}{\bullet}.$$

In the same vein, here is a list of more symmetries of the three dots:

$$\begin{gathered} \mathbf{I} \\ [1 \leftrightarrow 2] \\ [1 \leftrightarrow 3] \\ [2 \leftrightarrow 3] \\ [1 \to 2 \to 3 \to] \\ [1 \to 3 \to 2 \to] \end{gathered}$$

Is this list complete? The answer is "yes". Here's how to tell. Applying each symmetry to the colored dots $\textcolor{red}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}$ produces a new way of coloring the dots. If two symmetries produce the same coloring, they have the same effect on the object (the uncolored dots) and so are considered to be the same symmetry. Given the initial coloring of the dots, no two of the symmetries in the list above produce the same coloring. All those symmetries are, therefore, different. But is the list complete? We still haven't answered that question. To do so, we'll argue that there are exactly 6 symmetries of $\bullet\bullet\bullet$. Since we have six distinct symmetries in our list, our list must be complete.

Each symmetry produces a unique coloring of the dots (given the initial coloring: ${\color{red}\bullet}{\color{green}\bullet}{\color{blue}\bullet}$. There are three ways of coloring the first dot, two ways of coloring the second, and one way of coloring the third. Thus there are six total ways of coloring the dots and, therefore, six total symmetries. Thus our list is complete and no symmetry is listed more than once.

We can now make up a group table for $\mathbb{G}$. To do so, we go through a process similar to what we did for the symmetries of the square. For example, to compute

$$[1 \leftrightarrow 2] \circ [1 \to 2 \to 3 \to].$$

Look at what it does to the colors:

$$[1 \leftrightarrow 2] \circ [1 \to 2 \to 3 \to]({\color{red}\bullet}{\color{green}\bullet}{\color{blue}\bullet}) = [1 \leftrightarrow 2]({\color{blue}\bullet}{\color{red}\bullet}{\color{green}\bullet}) = {\color{red}\bullet}{\color{blue}\bullet}{\color{green}\bullet}.$$

Notice that this is the same coloring as the one given by $[2 \leftrightarrow 3]$:

$$[2 \leftrightarrow 3]({\color{red}\bullet}{\color{green}\bullet}{\color{blue}\bullet}) = {\color{red}\bullet}{\color{blue}\bullet}{\color{green}\bullet}.$$

Thus,

$$[1 \leftrightarrow 2] \circ [1 \to 2 \to 3 \to] = [2 \leftrightarrow 3].$$

Challenge! Find a more efficient way of computing the effect of combining two symmetries of $\bullet\bullet\bullet$.

For the complete group table for $\mathbb{G}$, see Table 2.

Some groups are so common that they deserve special names. Let D_n denote the symmetry group of a regular n–gon. Thus, the symmetry group of the square (which we studied previously) is denoted D_4. The symmetry group of n indistinguishable dots is denoted $\mathbb{S}_n$. Thus, the symmetry group of three indistinguishable dots (which we just studied) is denoted $\mathbb{S}_3$.

Exercise 4. (a) Show that every symmetry of 3 indistinguishable dots is also a symmetry of an equilateral triangle.

(b) Show that every symmetry of an equilateral triangle is also a symmetry of 3 indistinguishable dots.

(c) Explain why the previous two exercises show that D_3 is the same as $\mathbb{S}_3$.

(d) Show that D_n contains $2n$ symmetries.

(e) Show that $\mathbb{S}_n$ contains $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ symmetries.

(f) Explain why D_n is not the same as $\mathbb{S}_n$ for $n \geq 4$.

S_1

S_2

$S_2 \circ S_1$	$\mathbf{I}$	$[1 \leftrightarrow 2]$	$[1 \leftrightarrow 3]$	$[2 \leftrightarrow 3]$	$[1 \to 2 \to 3 \to]$	$[1 \to 3 \to 2 \to]$
$\mathbf{I}$	$\mathbf{I}$	$[1 \leftrightarrow 2]$	$[1 \leftrightarrow 3]$	$[2 \leftrightarrow 3]$	$[1 \to 2 \to 3 \to]$	$[1 \to 3 \to 2 \to]$
$[1 \leftrightarrow 2]$	$[1 \leftrightarrow 2]$	$\mathbf{I}$	$[1 \to 3 \to 2 \to]$	$[1 \to 2 \to 3 \to]$	$[2 \leftrightarrow 3]$	$[1 \leftrightarrow 3]$
$[1 \leftrightarrow 3]$	$[1 \leftrightarrow 3]$	$[1 \to 2 \to 3 \to]$	$\mathbf{I}$	$[1 \to 3 \to 2 \to]$	$[1 \leftrightarrow 2]$	$[2 \leftrightarrow 3]$
$[2 \leftrightarrow 3]$	$[2 \leftrightarrow 3]$	$[1 \to 3 \to 2 \to]$	$[1 \to 2 \to 3 \to]$	$\mathbf{I}$	$[1 \leftrightarrow 3]$	$[1 \leftrightarrow 2]$
$[1 \to 2 \to 3 \to]$	$[1 \to 2 \to 3 \to]$	$[1 \leftrightarrow 3]$	$[2 \leftrightarrow 3]$	$[1 \leftrightarrow 2]$	$[1 \to 3 \to 2 \to]$	$\mathbf{I}$
$[1 \to 3 \to 2 \to]$	$[1 \to 3 \to 2 \to]$	$[2 \leftrightarrow 3]$	$[1 \leftrightarrow 2]$	$[1 \leftrightarrow 3]$	$\mathbf{I}$	$[1 \to 2 \to 3 \to]$

TABLE 2. The group table for $\mathbb{S}_3$.