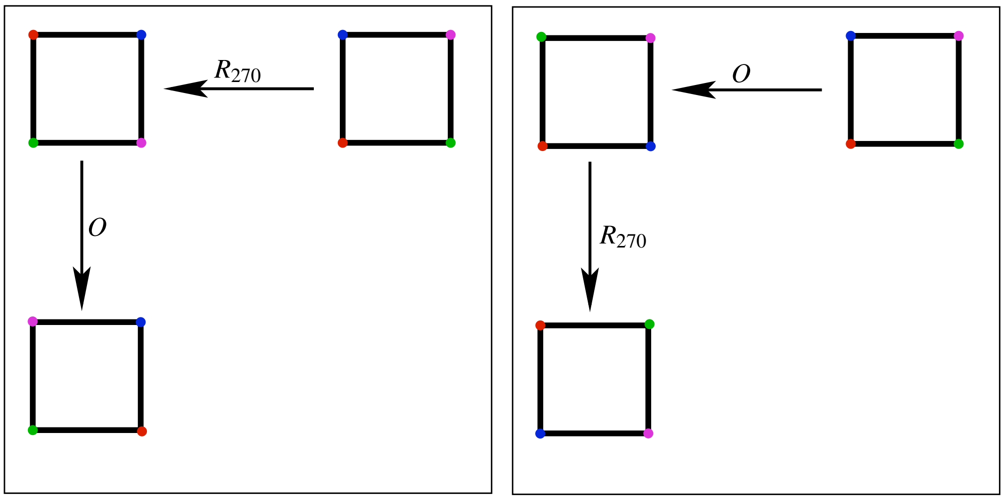
Now we can work on filling in the rest of the table. For example, to calculate $O \circ R_{270}$ we remember that this means that we rotate the square by 270° and then reflect over the off-diagonal axis. The left side of Figure 3 shows this operation. By examining the dots we see that $O \circ R_{270} = V$. The right side of Figure 3 shows that $R_{270} \circ O = H$. Notice that this means that

$$O \circ R_{270} \neq R_{270} \circ O.$$



$O \circ R_{270} = V$ $\qquad$ $R_{270} \circ O = H$

FIGURE 3. Calculating $O \circ R_{270}$ and $R_{270} \circ O$

Similar calculations allow us to fill in the rest of the table. See Table 1.

S_1

	$S_2 \circ S_1$	**I**	R_{90}	R_{180}	R_{270}	D	V	O	H
	I	**I**	R_{90}	R_{180}	R_{270}	D	V	O	H
	R_{90}	R_{90}	R_{180}	R_{270}	**I**	H	D	V	O
	R_{180}	R_{180}	R_{270}	**I**	R_{90}	O	H	D	V
S_2	R_{270}	R_{270}	**I**	R_{90}	R_{180}	V	O	H	D
	D	D	V	O	H	**I**	R_{90}	R_{180}	R_{270}
	V	V	O	H	D	R_{270}	**I**	R_{90}	R_{180}
	O	O	H	D	V	R_{180}	R_{270}	**I**	R_{90}
	H	H	D	V	O	R_{90}	R_{180}	R_{270}	**I**

TABLE 1. The group of symmetries of a square

Question: What patterns do you notice in the table?

> Possible patterns include:
> - ∘ every symmetry appears exactly once in each row and column.
> - ∘ performing a reflection and then another reflection is the same as performing a rotation.
> - ∘ For each symmetry, there is another symmetry which "undoes it".

The symmetries of the square are an example of what mathematicians call a **group**.

Definition 1. A **group** consists of a set G and an operation $\circ$ which combines two elements of G into a third element of G. That is, if a and b are in G then $a \circ b$ is also in G. Furthermore, we require the following properties to hold:

- (Associative) For any three elements a, b, c in G,
$$a \circ (b \circ c) = (a \circ b) \circ c.$$
- (Identity) There exists an element $\mathbf{I}$ in G (called the **identity element**) so that for every a in G,
$$a \circ I = I \circ a = a.$$
- (Inverses) For each a in G there exists some b in G so that
$$a \circ b = b \circ a = \mathbf{I}.$$
The element b is called the **inverse** of a and is sometimes written a^{-1}.

It need not be the case that for all a and b in X, $a \circ b = b \circ a$. That is, the group is not necessarily commutative. Indeed, the symmetries of the square are not commutative. Here is a fundamental observation which allows us to apply mathematics to the study of symmetry:

Theorem 1. For any object X, the set of symmetries of the object form a group. We denote the group $\texttt{Sym}(X)$. The operation is simply: first do one symmetry and then do another symmetry.

Exercise 1. For the group of symmetries of the square do the following:

- Pick three elements a, b, c at random and show that
$$a \circ (b \circ c) = (a \circ b) \circ c.$$
- Explain how the fact that an identity exists shows up in the table.
  The first row and the first column are exactly the same as the header column and the header row.
- Explain how the fact that each symmetry has an inverse symmetry shows up in the table.
  Every row and every column contains **I**.

We will occasionally make use of the following terminology:

- The **integers** are the positive and negative whole numbers and zero:
$$\{\dots, -2, -1, 0, 1, 2, \dots\}.$$
  They are denoted by $\mathbb{Z}$.
- The **rational numbers** are all the numbers which can be written as fractions of two integers. $\frac{1}{2}$, 3.0000009, and -17 are examples of rational numbers. The set of rational numbers is denoted $\mathbb{Q}$.
- The set of **real numbers** is the set of all numbers on the number line. It includes the integers and rational numbers as well as other numbers like $\sqrt{2}$ and π. The set of real numbers is denoted $\mathbb{R}$.
- The set of all real numbers except for zero is denoted $\mathbb{R}^*$.

Exercise 2. Decide whether or not the following are groups.

(a) $\mathbb{Z}$ with the operation of $+$.
(b) $\mathbb{Z}$ with the operation of $-$.
(c) $\mathbb{R}$ with the operation of $+$.
(d) $\mathbb{R}$ with the operation of $\cdot$ (multiplication).
(e) $\mathbb{R}^*$ with the operation of $\cdot$.

(a), (c), and (e) are groups. (b) is not a group because subtraction is not associative. (d) is not a group because 0 does not have a multiplicative inverse. (There is no number x so that $0 \cdot x = 1$.)

Notice that the groups in the previous exercise are not described as the symmetries of an object. A common philosophy in mathematics is: If you want to study an object, study its group of symmetries; if you want to study a group find an object for which the group is a group of symmetries.

Exercise 3. Show that the rotations of the square form a group. (Consider **I** to be a rotation.)