Fall 2022/MA 314 Final Takehome Exam

The Rules: You have from the time you receive the exam until 5 PM on Monday, December 19 to complete the exam and turn in your answer. Solutions should be stapled to the exam, have your name on every sheet, and either handed to Scott in person or slipped under his door.

You may use your textbooks and class notes, but you may not use any other resources in print or online. You may not talk about the exam with anyone except the professor. You are welcome to ask questions of clarification of your professor without cost (either by email or in person). You may also ask for hints (either by email or in person.) However, each hint provided will result in a grade reduction of 5% of the total possible points on the grade *for the problem for which the hint was requested.* Thus, on a problem if you have asked for and received two hints and then write a perfect solution your grade *for that problem* would be 18/20 = 90%. Thus, it may be worth it to ask for hints.

Sign here before completing the exam to indicate that you have read and abided by the rules:

A word of advice: Start this early. Don't turn in your first drafts. Turn in beautiful, readable, clear work.

There are two sections to this exam. Throughout the exam $\mathbb{H}^2$ refers to the upper half plane model of the hyperbolic plane and $\mathbb{B}$ refers to the disc model.

1. Explanations of things past

The following problems are all based on proofs covered in class or in the reading. Your solutions should clearly demonstrate your own mastery of the material (i.e. don't just copy from Bonahon or your notes, though as stated above you are allowed to use them as references.) **Do all of the following 3 problems.**

(Remark: "Detailed outline" means that with some concerted effort someone could come up with the complete proof of the result. The main details are provided and minor details are left out. Figuring out which is which is part of the task for this problem.)

(1) State the definition of the grasshopper metric and write 2-3 paragraphs explaining what it is, why we define it, and its basic properties. Be sure to discuss why we need such a complicated definition in general, but also why if we are working in the context of a group acting discontinuously on a metric space things are much simpler.

(2) Give a detailed outline of the proof that if X is a polygon in $\widetilde{X} \in \{\mathbb{E}^2, \mathbb{H}^2, \mathbb{S}^2\}$ with edge gluing isometries, then X tiles $\widetilde{X}$, if $\overline{X}$ is complete.

(3) Show that if $f(z) = \frac{az+b}{cz+d}$ is a linear fractional transform with a,b,c,d complex numbers such that $ad - bc = 1$, then f can be used to uniquely define an isometry of $\mathbb{H}^3$. Explain the relevance to showing that the group of orientation preserving isometries of $\mathbb{H}^3$ is isomorphic to $PSL(2,\mathbb{C})$ (the group of 2×2 matrices with complex entries subject to the rule that A is equivalent to $-A$ for each such matrix.) Hint: Consider Section 9.3 of Bonahon.

2. Some problems.

Do one of the following problems. Extra-credit will be given if you do more than one successfully.

(1) Suppose that Γ is a group acting discontinuously by isometries on a metric space X. Let $\overline{X}$ be the quotient space, with the grasshopper metric. (Recall that points $a,b \in X$ are equivalent if there is an element $g \in \Gamma$ such that $g(b) = a$ and that $\bar{a}$ is the equivalence class of a.) Let $\pi\colon X \to \overline{X}$ be the quotient map.

Let $a \in X$. Consider a path $\gamma\colon [0,1] \to \overline{X}$, with $\gamma(0) = \bar{a} \in \overline{X}$. Prove there exists a path $\tilde{\gamma}\colon [0,1] \to X$ such that $\tilde{\gamma}(0) = a$ and for all $t \in [0,1]$, $\pi(\tilde{\gamma}(t)) = \gamma(t)$.

You may use the following fact:

- Suppose that for each $t \in [0,1]$, $\varepsilon(t) > 0$. Then there exist finitely many $t_1, t_2, \ldots, t_n$ such that for all $t \in [0,1]$, $\gamma(t) \in \bigcup_{i=1}^{n} \overline{B}(\gamma(t_i), \varepsilon(t_i))$.

Hint: Use Bonahon Theorem 7.8 and define $\tilde{\gamma}$ inductively.

(Context: This theorem is central to showing that if the action is free and X is simply connected then Γ is isomorphic to the group of based loops in $\overline{X}$.)

(2) Do exercise 10.3 of Bonahon.

(3) Let $S \subset \mathbb{H}^3$ be a horosphere. (A **horosphere** is either a horizontal plane or a euclidean sphere in $\mathbb{R}^3$ tangent to the xy-plane and lying above the plane. The point of tangency is not considered to be part

of the horosphere.) Let d be the hyperbolic metric on $\mathbb{H}^3$ and let d_S be the path metric on S induced by d. That is, the length of a path in S is just it's hyperbolic length and if $x, y \in S$, $d_S(x,y)$ is the infimum of the lengths of paths in S joining x to y. Let d_{eucl} be the euclidean metric on $\mathbb{E}^2$. Prove that there is a constant C (depending on S) such that (S, d_S) is isometric to $(\mathbb{E}^2, Cd_{\text{eucl}})$.

(4) Here is another kind of 3-dimensional geometry, denoted $\mathbb{R}\times\mathbb{H}^2$. The set is the same as for $\mathbb{H}^3$: $\{(x,y,u) : u > 0\}$. For a point $(x,y,u) \in \mathbb{R}\times\mathbb{H}^2$, the tangent space $T_{(x,y,u)}$ is a 3-dimensional vector space equivalent to $\mathbb{R}^3$. The vector (a,b,c) can be thought of as an arrow based at (x,y,u) and pointing a units along the x-axis, b units along the y-axis, and c-units along the z-axis.

For $\vec{v} = (a,b,c) \in T_{(x,y,u)}$ and $\vec{w} = (d,e,f) \in T_{(x,y,u)}$ define

$$\langle \vec{v}, \vec{w} \rangle_{(x,y,u)} = ad + \frac{be + cf}{u^2}$$

Notice that

$$\langle \vec{v}, \vec{w} \rangle_{(x,y,u)} = a \cdot d + \frac{(b,c)\cdot(e,f)}{u^2}$$

where the second occurence of $\cdot$ is the dot product on $\mathbb{R}^2$ and the first is usual multiplication in $\mathbb{R}$ (i.e. the one-dimensional dot product).

For a smooth path $\gamma\colon\ [t_0, t_1]$ in $\mathbb{R}\times\mathbb{H}^2$, define

$$L(\gamma) = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}^{1/2}\, dt.$$

As usual, for points $z, w \in \mathbb{R}\times\mathbb{H}^2$ define $d(z,w)$ to be the infimum of the lengths of all piecewise C^1 paths joining z to w.

(a) Prove that $\langle\, , \rangle_{(x,y,u)}$ is an inner product on $T_{(x,y,u)}$. Since the formula is continuous in (x,y,u), this defines a Riemannian metric on $\mathbb{R}\times\mathbb{H}^2$. (Riemannian metrics are defined in Schwartz only for subsets of $\mathbb{R}^2$, but the definition is easily extended to this setting.)

(b) Prove that if $\phi:\ \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry then the function

$$\widehat{\phi}\colon\ \mathbb{R}\times\mathbb{H}^2 \to \mathbb{R}\times\mathbb{H}^2$$

defined by $\widehat{\phi}(x,(y,u)) = (x, \phi(y,u))$ is an isometry.

(c) Prove that for all $x_0 \in \mathbb{R}$, the function

$$\psi\colon\ \mathbb{R}\times\mathbb{H}^2 \to \mathbb{R}\times\mathbb{H}^2$$

defined by $\psi(x,y,u) = (x + x_0, y, u)$ is an isometry.

(d) Prove that $\mathbb{R}\times\mathbb{H}^2$ is homogeneous.

(e) Prove that if $(x,y,u), (x',y,u) \in \mathbb{R}\times\mathbb{H}^2$ with $x \leq x'$, then the straight line

$$\gamma(t) = (t, y, u)$$

for $t \in [x, x']$ is a geodesic in $\mathbb{R}\times\mathbb{H}^2$.

(f) Prove that if $(x_0, y, u), (x_0, y', u') \in \mathbb{R}\times\mathbb{H}^2$ then the geodesic from (x_0, y, u) to (x_0, y', u') is either a vertical line segment in the half plane $x = x_0$ or a portion of a circle in the plane $x = x_0$ whose center is on the xy-plane. (You may use previous results.)

(g) Prove that for all $z, w \in \mathbb{R}\times\mathbb{H}^2$ if $d(z,w) = 0$ then $z = w$. (Hint: consider the projection map $p\colon\ \mathbb{R}\times\mathbb{H}^2 \to \mathbb{H}^2$ defined by $p(x,y,u) = (y,u)$. Show that it is distance non-increasing and then cleverly combine results concerning $\mathbb{H}^2$ and previous parts of this problem.)