Fall 2022/MA 314 Study Guide for Exam 2

This is an optional exam. If you decide to take it, you need to schedule 50 minutes during the week of November 7 to take the exam at Scott's office. If you received a grade of $X/100$ on Exam 1 and a grade of $Y/100$ on Exam 2. Your Exam 1 grade will be revised to be $(X+(Y/100)*(100-X))/100$.

The exam will consist of a selection of the following, perhaps phrased in slightly different ways.

(1) Know precise definitions of the following terms:

  (a) Metric space (X,d)
  (b) Discrete, Euclidean, Manhattan, Paris, Max, Spherical, Hyperbolic metrics
  (c) Isometry $f\colon X \to Y$ between metric spaces
  (d) Open ball and open set in a metric space
  (e) Continuous function $f\colon X \to Y$ between metric spaces
  (f) Topologically continuous function $f\colon X \to Y$ between metric spaces
  (g) The limit (x_n) of a sequence in a metric space, and what it means for a sequence to converge.
  (h) Sequentially continuous function $f\colon X \to Y$ between metric spaces
  (i) Homeomorphism $f\colon X \to Y$ between metric spaces
  (j) Compact metric space
  (k) The length of a sequence in a metric space
  (l) Complete metric space
  (m) Surface
  (n) Linear and Anti-linear fractional transform
  (o) Grasshopper metric
  (p) "Locally isometric"

(2) Be able to prove the following:

  (a) A function $f\colon X \to Y$ between metric spaces is continuous iff it is topologically continuous iff it is sequentially continuous.
  (b) If $f\colon X \to Y$ is a surjective continuous function and X is compact, then Y is compact
  (c) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then so is $g \circ f\colon X \to Z$
  (d) If a finite length sequence in a metric space X has a convergent subsequence, then it also converges.
  (e) If X is a compact metric space, then it is complete.
  (f) Suppose that $U \subset \mathbb{E}^n$ has the property that there is a piecewise differentiable path in U between any two points of U. Define the path metric on U and prove that it is a metric.
  (g) Prove that every isometry of $\mathbb{E}^2$ is the composition of translations, reflections, and rotations. If there are lemmas you need along the way, be able to state and prove them too.
  (h) Prove that if τ and τ' are reflections of $\mathbb{E}^2$ about lines through the origin, then $\tau \circ \tau'$ is a rotation of $\mathbb{E}^2$ about the origin.

(i) Prove that every geodesic in $\mathbb{E}^2$ is a line segment.

(j) Prove that if ρ and ρ' are rotations of $\mathbb{E}^3$ around lines through the origin, then so is $\rho' \circ \rho$.

(k) Prove that horizontal translations, homotheties, and the standard inversion are all isometries of $\mathbb{H}^2$.

(l) Prove that every geodesic in $\mathbb{H}^2$ is either a vertical line segment or a portion of a circle centered on the x-axis in $\mathbb{E}^2$.

(m) Prove that if $a, b \in \mathbb{R} \cup \{\infty\}$ are distinct then there exists a hyperbolic isometry ϕ taking the complete geodesic with ends at a and b to the complete geodesic with ends at 0 and ∞ (i.e. the positive imaginary axis.)

(n) Prove that if X is a Euclidean polygon and that $\sim$ is an equivalence relation induced by gluing pairs of edges via isometries with quotient space $\overline{X}$, then the gluing is proper and for every $\overline{P} \in \overline{X}$ there exists $r > 0$ such that $B_r(\overline{P}) = \bigcup_{x \in \overline{P}} B(x, r)$.