Why does math work the way it does? Does it have to work that way? The following questions ask you to explore variations on elementary school mathematics. For each question, take some time to read it, then share some ideas about how to answer it. One person in the group should act as a note-taker and share their notes with everyone, so each person can reflect on their own about the project.

You are not required to get through all these questions. If you get stuck on a problem, move on to later problems. The write-up should be done outside of class, but otherwise this is not out-of-class work. It's a chance to do some mathematical investigation, meet your classmates, and begin to communicate mathematically. The instructions for the write-up are at the end.

Throughout we'll work only with the **integers**. These are the positive and negative whole numbers, and zero. We denote them as follows:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

For a fixed integer n, we let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the nonnegative integers less than n.

(1) As you know, we can add, subtract, and multiply integers. What does multiplication mean? Why is $(-1)(-1) = +1$?

(2) What is division of integers? How does it relate to multiplication? When can we divide two integers and still get an integer?

(3) As carefully as you can, craft a sentence saying *exactly* what it means for an integer a to be a "multiple" of an integer b. Compare sentences. As you think about your group members' sentences, can you think of some examples for a and b to test their definition? Do you agree that their definition captures the exact meaning of the word? Why or why not? Can you agree on a definition that does not involve division?

(4) Recall that if a and $b > 0$ are integers, we can divide a by b to get a "quotient" and a "remainder". For example, 13 divided by 5 has a quotient of 2 and a remainder of 3. Why is the remainder when dividing by 5 one of the numbers in $\mathbb{Z}_5$? More generally, why is the remainder when dividing by b one of the numbers in $\mathbb{Z}_b$?

(5) Write an equation using only integers to express the statement: "the remainder of a when dividing by b is r" (Hint: you'll also have to use the quotient) Be sure to say what your variables mean.

(6) For the purposes of this problem, let $[a]_b$ denote the remainder of a when you divide by b (We'll always require that $b \geq 1$.) For example $[13]_5 = 3$.

  (a) What is $[8+32]_7$? Is it equal to $[8]_7 + [32]_7$?
  (b) Is $[x]_b + [y]_b$ equal to $[x+y]_b$ no matter what x, y, b are? Why or why not?

(7) For a fixed $b \geq 1$, define a new kind of addition (call it "diamond addition"!) by:

$$[x]_b \diamond_b [y]_b = [x+y]_b$$

It is defined on numbers in $\mathbb{Z}_b$.

  (a) Summarise in words what this definition is saying.
  (b) Explain why this addition always produces a number in $\mathbb{Z}_b$.

(c) Is this addition commutative? (That is, is $x \diamond_{b} y = y \diamond_{b} x$ no matter what x, y, b are?) Why or why not?
(d) Can you find a number y in $\mathbb{Z}_{12}$ so that $9 \diamond_{12} y = 7$?
(e) For a given x is it always possible to find a number y in $\mathbb{Z}_{12}$ so that $x \diamond_{12} y = 7$? Why or why not?
(f) Notice that if a and c have the same remainder when we divide by b, then $[a]_b = [c]_b$. If $[a_1]_b = [c_1]_b$ and $[a_2]_b = [c_2]_b$, is it true that

$$[a_1]_b \diamond_{b} [a_2]_b = [c_1]_b \diamond_{b} [c_2]_b?$$

(8) For a fixed $b \geq 1$, define a new kind of multiplication by:

$$[x]_b \odot_{b} [y]_b = [x \cdot y]_b$$

(a) Compute $[8]_5 \odot_{5} [7]_5$.
(b) Summarise in words what this definition is saying.
(c) Explain why this multiplication always produces a number in $\mathbb{Z}_b$.
(d) Is this multiplication commutative? (That is, is $x \odot_{b} y = y \odot_{b} x$ no matter what x, y, b are?) Why or why not?
(e) If $[a_1]_b = [c_1]_b$ and $[a_2]_b = [c_2]_b$, is it true that

$$[a_1]_b \diamond_{b} [a_2]_b = [c_1]_b \diamond_{b} [c_2]_b?$$

(f) Can you find a number y in $\mathbb{Z}_{12}$ so that $7 \odot_{12} y = 1$? Why or why not?
(g) Can you find a number y in $\mathbb{Z}_{12}$ so that $3 \odot_{12} y = 1$? Why or why not?
(h) Can you articulate a general principle about numbers x, y, z in $\mathbb{Z}_{12}$ such that $x \odot_{12} y = z$? What if 12 is changed to some other number?

(9) For a fixed $b \geq 1$, define a new kind of exponentiation by

$$[x]_b \star_{b} [y]_b = [x^y]_b$$

(a) Compute $[8]_5 \star_{5} [7]_5$.
(b) Compute $[3]_5 \star_{5} [2]_5$.
(c) Summarise in words what this definition is saying
(d) Explain why this exponentiation always produces a number in $\mathbb{Z}_b$.
(e) If $[a_1]_b = [c_1]_b$ and $[a_2]_b = [c_2]_b$, must it be true that

$$[a_1]_b \star_{b} [a_2]_b = [c_1]_b \star_{b} [c_2]_b?$$

(10) Project Summary: Write a few short paragraphs answering the following. This is what you will turn in. Be sure to put your name on it!

(a) Who was in your group?
(b) How well did your group function together? Did everyone share ideas? Did you listen well to each other? Do you feel like conversation was productive?
(c) Which problems did you have the easiest time with? Which were the most challenging?