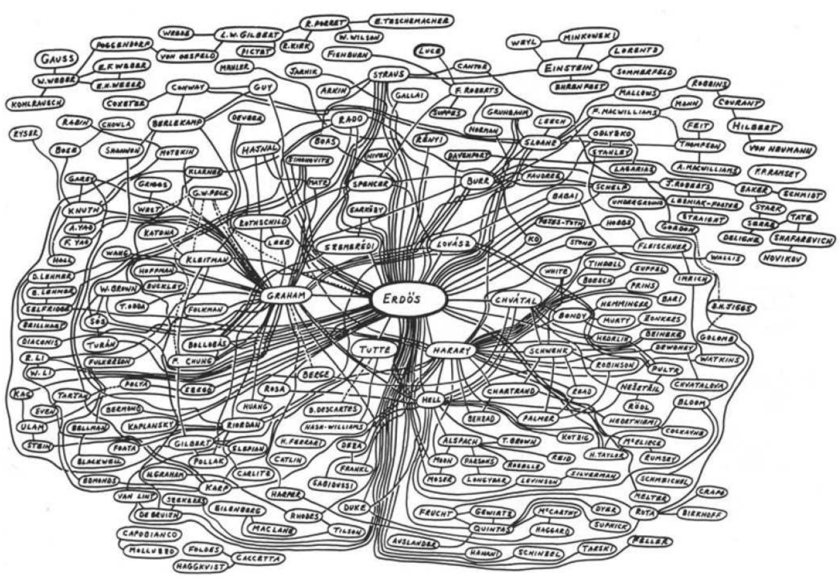
Fall 2022/MA 274 GP 2: Networks

You are not required to get through all these questions. The write-up should be done outside of class, but otherwise this is not out-of-class work. It's a chance to do some mathematical investigation, meet your classmates, and begin to communicate mathematically. The instructions for the write-up are at the end.

The word "graph" in mathematics has multiple meanings. One meaning is the "graph of a function," as in Calculus. Another meaning, and the one we'll use for this project, is that of a network, consisting of nodes and connections between the nodes. For example, there is the Facebook graph where each node (or **vertex**) is a Facebook user and there is a connection between them if they are "friends." Similarly, there is the collaboration graph in mathematics. Here is a small portion of it, involving the famous mathematician Paul Erdös[1]. Each node is a mathematician and two nodes are joined if the mathematicians co-authored a paper.

Once again at this point in the semester you won't be able to provide perfect answers! Don't worry about it, just do your best.

For each question, take some time to read it, then share some ideas about how to answer it. One person in the group should act as a note-taker and share their notes with everyone, so each person can reflect on their own about the project.



(1) Each member of your group should draw two graphs with 5 vertices (aka nodes) each (so if you have 3 group members you have 6 graphs to look at.) Compare your graphs. In what sense are they similar? In what sense are they fundamentally different? Is it possible for the same network to be drawn in two very different ways?

(2) Consider the network of social connections. If you had to get an urgent message to the governor of New Mexico by passing to someone you know reasonably well and having them pass it to someone they know reasonably well, etc. How would you get the message to her? What's the

[1]The image is from the blog "Not quite the economist" and was hand-drawn by the graph theorist Ron Graham.

fewest number of people it would need to pass through? The sequence of connections you use to get the message to the governor is an example of a **path** in a graph (aka network).

(3) Each person in your group should try to articulate as carefully as possible a definition for "path from vertex a to vertex b" in a graph. After crafting your definitions, compare them. Can you find an example of a graph so that there is a path that satisfies one person's definition but not another's? Settle on a single definition for your group to use for the remainder of this project. As you proceed, if you find you need to come back to this question and modify the definition, feel free to do so.

(4) The **length** of a path is just the number of edges it traverses. Draw a graph with at least 12 vertices and a large number of edges and give an example of a path in the graph of length at least 5. Use your careful definition to explain why this counts as a path. Also give an example of something that is not a path and use the definition to explain why it is not a path.

(5) Use your definition to give a careful argument that if, in a particular graph, there is a path from vertex a to vertex b and another path from vertex b to vertex c, then there must also be a path from vertex a to vertex c. Be sure your explanation explicitly references your precise definition.

(6) A graph is **connected** if for any two vertices in the graph there is always a path from one to the other. Give an example of a connected graph and give an example of a disconnected graph.

(7) Consider a graph (which could be either connected or disconnected). For vertices a and b, define the symbol $a \sim b$ to mean that there is a path from a to b. Explain why the following are true:

   (a) For every vertex a, $a \sim a$.

   (b) For any two vertices a and b, it is that case that if $a \sim b$ then also $b \sim a$

   (c) For any three vertices a, b, c, it is the case that if $a \sim b$ and $b \sim c$, then $a \sim c$.

(8) A **backtrack** in a path is when you traverse an edge and then immediately traverse it again in the opposite direction. Consider a graph (which could be either connected or disconnected). For any two vertices a and b, define the symbol $a \approx b$ to mean that there is a path from a to b that does not contain any backtracks and which has an even number of edges. Do your best to determine if the following must be true.

   (a) For every vertex a, it must be the case that $a \approx a$.

   (b) For any two vertices a and b, it is must be the case that if $a \approx b$ then also $b \approx a$

   (c) For any three vertices a, b, c, it must be the case that if $a \approx b$ and $b \approx c$, then $a \approx c$.

(9) For a particular connected graph, if a and b are any vertices of the graph, define the **distance** from a to b to be the *minimum* length of a path from a to b. Call this distance $d(a,b)$ Return to your graph with 12 or more vertices and for different choices of vertices, calculate the distance between them.

(10) Using your precise definition of path, give (as much a possible) careful explanations of the following facts:

   (a) For every vertex a, $d(a,a) = 0$.
   (b) For any two vertices a, b, if $d(a,b) = 0$ then $a = b$.
   (c) For every pair of vertices a, b we must have $d(a,b) = d(b,a)$.
   (d) For every triple of vertices a, b, c, we must have $d(a,c) \leq d(a,b) + d(b,c)$.

(11) (Food for thought!) Consider one of the (virtual or non-virtual) social networks you are a part of. Explain why it is likely that most of the people you are directly connect to are themselves directly connected to more people than you are.

(12) Project Summary: Write a few short paragraphs answering the following. This is what you will turn in. Be sure to put your name on it!

   (a) Who was in your group?
   (b) How well did your group function together? Did everyone share ideas? Did you listen well to each other? Do you feel like conversation was productive?
   (c) Which problems did you have the easiest time with? Which were the most challenging?