MA 274: Exam 2 Study Guide Partial Solutions

(1) The following are examples of equivalence relations:

- $\equiv_p$ on $\mathbb{Z}$, where $x \equiv_p y$ iff $x - y$ is a multiple of p.

  Proof. We show that $\equiv_p$ is reflexive, symmetric, and transitive. Suppose that $n \in \mathbb{Z}$. Then

  $$n - n = 0 = 0 \cdot p$$

  so $n \equiv_p n$. Thus, $\equiv_p$ is reflexive.

  Suppose that $n, m \in \mathbb{Z}$. We prove that if $n \equiv_p m$ then $m \equiv_p n$. Assume $n \equiv_p m$. By the definition of $\equiv_p$, there exists $k \in \mathbb{Z}$ such that

  $$n - m = kp.$$

  Thus, by algebra,

  $$m - n = (-k)p$$

  Since $-k \in \mathbb{Z}$, $m \equiv_p n$. Thus, $\equiv_p$ is symmetric.

  Suppose that $n, m, \ell \in \mathbb{Z}$. We prove that if $n \equiv_p m$ and $m \equiv_p \ell$, then $n \equiv_p \ell$. Assume $n \equiv_p m$ and $m \equiv_p \ell$. By the definition of $\equiv_p$, there exists, $x, y \in \mathbb{Z}$ such that

  $$\begin{aligned} n - m &= xp \text{ and} \\ m - \ell &= yp \end{aligned}$$

  Adding our two equations we obtain

  $$n - \ell = (x + y)p.$$

  Since $x + y \in \mathbb{Z}$, we have $n \equiv_p \ell$ and so $\equiv_p$ is transitive. □

- $\sim$ on $\mathbb{Z} \times \mathbb{N}$ where $(a, b) \sim (c, d)$ iff $ad = bc$.

  Proof. We prove $\sim$ is reflexive, symmetric, and transitive. Let $(a, b) \in \mathbb{Z} \times \mathbb{N}$. Then since $ab = ba$, $(a, b) \sim (a, b)$ so $\sim$ is transitive. Suppose that $(a, b) \sim (c, d)$. Then by definition of $\sim$, $ad = bc$. By the properties of multiplication, $cb = da$, so $(c, d) \sim (a, b)$. Thus, $\sim$ is symmetric. Finally, assume that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. By definition of $\sim$, $ad = bc$ and $cf = ed$. Multiplying the first equation by f produes:

  $$adf = bcf.$$

  Use the second equation to subsititue $cf = ed$:

  $$adf = bed.$$

  Since $d \neq 0$, we can divide both sides by d to find $af = be$. Thus, $(a, b) \sim (e, f)$. Thus, $\sim$ is transitive.

  □

- $\sim$ on the vertex set of a graph G, where $x \sim y$ iff there is a path from x to y in the graph.

Proof. We prove $\sim$ is reflexive, symmetric, and transitive. Suppose that x is a vertex of G. Then (x) is a finite sequence whose initial term is x and whose final term is x and with the (vacuous) property that each pair of adjacent vertices in the sequence are the endpoints of an edge in G. Thus, (x) is a path from x to x and so $\sim$ is reflexive.

Suppose that x and y are vertices of G such that $x \sim y$. We show $y \sim x$. By definition of $\sim$, there is a path

$$x_0, x_1, \dots, x_n$$

from $x = x_0$ to $y = x_n$. By definition of path each pair x_i, x_{i+1} of adjacent vertices are the endpoints of an edge in G. For each $i \in \{0, \dots, n\}$, let $y_i = x_{n-i}$. Then

$$y_0, y_1, \dots, y_n$$

is the sequence $x_0, \dots, x_n$ in reverse order. Thus, $y_0 = x_n = y$ and $y_n = x_0 = x$. Also, if y_i and y_{i+1} are adjacent vertices, then they equal x_{n-i-1} and x_{n-i} respectively and so are adjacent vertices in the path from x to y. Consequently, they are the endpoints of an edge in G. Thus, y_i and y_{i+1} are endpoints of an edge in G. Thus, we have a path from y to x. Consequently, $y \sim x$.

Now suppose that x, y, z are vertices of G and that $x \sim y$ and $y \sim z$. We show $x \sim z$. By definition of $\sim$, there is a path

$$x_0, x_1, \dots, x_n$$

from x to y and also a path

$$y_0, \dots, y_m$$

from y to z. Notice that $x_n = y = y_0$. We claim that the sequence

$$x_0, x_1, \dots, x_n, y_1, \dots, y_m$$

is a path from x to z. Recall that $x_0 = x$ and $y_m = z$. Any pair of adjacent vertices in the sequence is either x_i, x_{i+1} for $i \in \{0, \dots, n-1\}$ or y_i, y_{i+1} for $i \in \{1, \dots, m-1\}$ or x_n, y_1. In the first two cases, the adjacent vertices are endpoints of edges since they were in the paths from x to y and from y to z. In the third case we have $x_n = y_0$ so x_n and y_1 are also the endpoints of an edge in G. Consequently, there is a path from x to z. Thus, $x \sim z$ by definition of $\sim$ and so $\sim$ is transitive. $\square$

- Suppose that P is a partition of a non-empty set X. Define $\sim$ on X by $x \sim y$ iff there exists $A \in P$ such that x, y are both elements of A.

  Proof. We prove that $\sim$ is reflexive, symmetric and transitive. Let $x \in X$. By the covering property of partitions, there exists $A \in P$ such that $x \in A$. Thus, there exists $A \in P$ such that $x \in A$ and $x \in A$. Consequently, $x \sim x$. Thus, $\sim$ is reflexive.

  Now suppose that $x, y \in X$ and that $x \sim y$. By definition of $\sim$, there exists $A \in P$ such that $x \in A$ and $y \in A$. By the definition of conjunction, $y \in A$ and $x \in A$. Thus, $y \sim x$. Thus, $\sim$ is symmetric.

  Assume that $x, y, z \in X$, that $x \sim y$ and that $y \sim x$. Thus, there exists $A \in P$ and $B \in P$ so that $x, y \in A$ and $y, z \in B$ by definition of $\sim$. Observe that $y \in A \cap B$. Since $A \cap B \neq \varnothing$ we must have $A = B$ by the pairwise disjoint condition for partitions. Thus, $x, z \in B = A$. Therefore, $x \sim z$. So $\sim$ is transitive. $\square$

(2) Suppose that $\sim$ is an equivalence relation on a non-empty set X. For each $x \in X$, let $[x]$ be the equivalence class of x. Then the following hold:

(a) For all $x \in X$, $x \in [x]$.
(b) For all $x, y \in X$, $x \sim y$ iff $[x] = [y]$.
(c) For all $x, y \in X$, $[x] \cap [y] \neq \varnothing$ implies $[x] = [y]$.

Proof. Assume that $\sim$ is an equivalence relation. Recall that $[x] = \{y \in X : x \sim y\}$. Since $\sim$ is reflexive, $x \sim x$. Thus, $x \in [x]$.

Suppose that $x \sim y$. We show that $[x] \subset [y]$. Let $z \in [x]$. Thus, $x \sim z$ by definition of equivalence class. Since $\sim$ is symmetric, $y \sim x$. Since $\sim$ is transitive, $y \sim z$. Thus, $z \in [y]$. Since z was arbitrary, $[x] \subset [y]$. The same proof applied to the expression $y \sim x$ (since $\sim$ is symmetric) shows that $[y] \subset [x]$. Thus, $[x] = [y]$. Thus, if $x \sim y$, then $[x] = [y]$.

Now suppose that $[x] = [y]$. By the first, part $y \in [y]$. By the definition of set equality, $y \in [x]$. By definition of equivalence class, $x \sim y$. Thus, $[x] = [y]$ implies $x \sim y$.

Finally, suppose that $x, y \in X$ and that $[x] \cap [y] \neq \varnothing$. We show $[x] = [y]$. Since $[x] \cap [y] \neq \varnothing$, there exists $z \in [x] \cap [y]$. By definition of intersection $z \in [x]$ and $z \in [y]$. By definition of equivalence class, $x \sim z$ and $y \sim z$. By the second part, $[x] = [z]$ and $[y] = [z]$. Thus, $[x] = [y]$. □

(3) If $\sim$ is an equivalence relation on a non-empty set X, then the set of equivalence classes is a partition of X.

Proof. From the theorem above, for every $x \in X$, $x \in [x]$ so $X/\sim$ satisfies the covering property. If $A \in X/\sim$ then by definition, there exists $x \in X$ such that $A = [x]$. Since $x \in [x]$, no element of $X/\sim$ is empty. If $A, B \in X/\sim$ and $A \cap B \neq \varnothing$, then $A = B$ by the final conclusion of the previous theorem. □

(4) If $f\colon \mathbb{Z}/\equiv_p \to \mathbb{Z}/\equiv_p$ is defined by $f([x]) = [2x]$ then f is well-defined.

Proof. Suppose that $[x] = [y]$. By the Fundamental Properties of Equivalence Relations, $x \equiv_p y$. By the definition of $\equiv_p$, there exists $k \in \mathbb{Z}$ such that $x = y + pk$. Then

$$2x = 2y + p(2k).$$

Since $2k \in \mathbb{Z}$, $2x \equiv_p 2y$. By the Fundamental Properties of Equivalence Relations, $[2x] = [2y]$. Thus, $f([x]) = f([y])$ and f is well-defined. □

(5) Addition and multiplication in $\mathbb{Z}/\equiv_p$ are well-defined.

Proof. We just prove that multiplication is well-defined. Suppose that $[x] = [x']$ and $[y] = [y']$. We need to show that $[x][y] = [x'][y']$.

Since $[x] = [x']$ by the Fundamental Properties of Equivalence Relations, $x \equiv_p x'$. Likewise, $y \equiv_p y'$. By definition of $\equiv_p$, there exists $k, \ell \in \mathbb{Z}$ so that

$$\begin{aligned} x &= x' + pk \\ y &= y' + p\ell. \end{aligned}$$

Multiplying we get:

$$xy = x'y' + p(\ell x' + ky' + pk\ell).$$

Thus, $xy \equiv_p x'y'$ and so $[x][y] = [xy] = [x'y'] = [x'][y']$ as desired. □

(6) The compositions of injections/surjections/bijections is a an injection/surjection/bijection.

Proof. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are functions.

Claim 1: If f and g are both injections, so is $g \circ f$.

Let $a, b \in X$ and assume that $g \circ f(a) = g \circ f(b)$. That is, $g(f(a)) = g(f(b))$. Since g is injective, $f(a) = f(b)$. Since f is injective, $a = b$. Thus, $g \circ f$ is also injective.

Claim 2: If f and g are both surjections, so is $g \circ f$.

Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective there exists $x \in X$ so that $f(x) = y$. Then

$$g \circ f(x) = g(f(x)) = g(y) = z.$$

Thus, $g \circ f$ is surjective.

Claim 3: If f and g are bijections, so is $g \circ f$.

This follows immediately from Claims 1 and 2 and the definition of bijection. □

(7) A function $f\colon X \to Y$ is a bijection if and only if there is a function $f^{-1}\colon Y \to X$ such that $f \circ f^{-1}(y) = y$ for all $y \in Y$ and $f^{-1} \circ f(x) = x$ for all $x \in X$.

Proof. Suppose, first, that $f\colon X \to Y$ is a bijection. Define $g\colon Y \to X$ as follows. If $f(x) = y$, let $g(y) = x$. We claim g is a function. Since f is surjective, for every $y \in Y$, there exists $x \in X$ with $f(x) = y$. Thus, g satisfies the domain condition. Suppose $y_1 = y_2$ and that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $f(x_1) = f(x_2)$, so since f is injective $x_1 = x_2$. Thus, $g(y_1) = x_1 = x_2 = g(y_2)$, so g satisfies the well-defined condition. We now claim that g is the inverse of f. Let $x \in X$ and let $y = f(x)$. Then $g \circ f(x) = g(f(x)) = g(y) = x$. The first equality is the definition of composition, the second is the definition of y, and the third is the definition of g. Similarly, suppose that $y \in Y$ and let $x = g(y)$. Then $f \circ g(y) = f(g(y)) = f(x) = y$, with the last equation coming from the definition of g and x.

Finally note that $g \circ f\colon X \to X$ by the definition of function composition, so it has the same domain and codomain as the identity function on X. Since it takes the same value on each element of X as does the identity function, it is equal to the identity function. Likewise $f \circ g\colon Y \to Y$ is equal to the identity function on Y.

Thus, g is the inverse function to f; traditionally denoted f^{-1}.

□

(8) The set of bijections from a set X to itself is a group, with function composition as the operation.

Proof. Let G be the set of bijections from X to itself with function composition as the operation. We just proved above that if f and g are bijections, so is $g \circ f$. Thus, G satisfies (G1). Set $\nVdash = \mathrm{id}_X$. Observe it is a bijection since it is it's own inverse function. If $f\colon X \to Y$ is a bijection, then

$$f\colon \nVdash(x) = f\colon \ \mathrm{id}_X(x) = f(\mathrm{id}_X(x)) = f(x)$$

and

$$\nVdash\colon f(x) = \mathrm{id}_X\colon f(x) = \mathrm{id}_X(f(x)) = f(x)$$

for all $x \in X$, by definition of id_X and function composition. Thus, the functions f and $f \circ \nVdash$ and $\nVdash \circ f$ are all equal. Thus, $\nVdash$ is a group identity element for G.

We proved above that if $f \in G$, then f has an inverse function f^{-1}. Since $f = (f^{-1})^{-1}$, f^{-1} has an inverse and is also a bijection. Hence, $f^{-1} \in G$. Also, by definition, $f\colon f^{-1} = \mathrm{id}_X = f^{-1}\colon f$ and so since $\nVdash = \mathrm{id}_X$, we have that f^{-1} is the group inverse for f.

Finally, it was proved in class that function composition is associative. Thus G is a group.

□

(9) The function $f\colon \mathbb{Z}/\equiv_{10} \to \mathbb{Z}/\equiv_{10}$ defined by $f([x]) = [2x]$ is not injective or surjective.

Proof. Observe that $f([0]) = [2 \cdot 0] = [0]$ and $f([5]) = [2 \cdot 5] = [10] = [0]$. Since $[0] \neq [5]$ in $\mathbb{Z}/\equiv_{10}$, f is not injective.

The function f is not surjective, for $[1] \notin \operatorname{range} f$. To see this, suppose (for a contradiction) that $f([x]) = [1]$ for some $[x] \in \mathbb{Z}/\equiv_{10}$. Then

$$[2x] = [1].$$

By the fundamental properties of equivalence relations and the definition of $\equiv_{10}$, there exists $k \in \mathbb{Z}$ so that

$$2x = 1 + 10k.$$

Thus,

$$1 = 2(5k - 1).$$

But 1 is not an even integer, and so we have a contradiction. □

(10) The function $f\colon \mathbb{Z}/\equiv_{10} \to \mathbb{Z}/\equiv_{10}$ defined by $f([x]) = [3x]$ is injective and surjective.

Proof. Suppose that $f([a]) = f([b])$. By definition of f,

$$[3a] = [3b].$$

By the fundamental properties of equivalence relations and the definition of $\equiv_{10}$,

$$3a = 3b + 10k$$

for some $k \in \mathbb{Z}$. Consequently,

$$3(a - b - 3k) = k$$

Set $m = (a - b - 3k)$, so $k = 3m$. Thus,

$$3(a - b) = 30m.$$

We see that

$$(a - b) = 10m.$$

Thus, $a \equiv_{10} b$ and so $[a] = [b]$. Thus, f is injective.

To see that $f\colon \mathbb{Z}/\equiv_{10} \to \mathbb{Z}/\equiv_{10}$ is surjective, let $[y] \in \mathbb{Z}/\equiv_{10}$. We must find an $[x] \in \mathbb{Z}/\equiv_{10}$ so that

$$[3x] = [y].$$

We could do this by just computing $f([x])$ for every $[x] \in \mathbb{Z}/\equiv_{10}$ and seeing that there will be an $[x]$ such that $f([x]) = [y]$ for every $[y]$. For fun, we take a slightly different approach.

Observe first that if $[a], [b] \in \mathbb{Z}/\equiv_{10}$ then

$$f([a]+[b]) = f([a+b]) = [3(a+b)] = [3a]+[3b] = f([a])+f([b]).$$

Thus, if $[w],[y] \in \operatorname{range} f$, then there exist $[a],[b] \in \mathbb{Z}/\equiv_{10}$ so that $f([a]) = [w]$ and $f([b]) = [y]$. In which case,

$$f([a]+[b]) = f([a])+f([b]) = [w]+[y],$$

so $[w]+[y] \in \mathbb{Z}/\equiv_{10}$.

Now notice that $f([7]) = [21] = [1]$. Now suppose that $[y] \in \mathbb{Z}/\equiv_{10}$. Without loss of generality, we may assume that $y \geq 0$ (otherwise replace y with $y+10|y|$). Then

$$[y] = \underbrace{[1]+\cdots+[1]}_{y \text{ times}}.$$

By our previous remarks, this means that

$$[y] = f\Big(\underbrace{[7]+\cdots+[7]}_{y \text{ times}}\Big)$$

Thus, f is surjective. □

(11) Define $\sim$ on $\mathbb{Z}\times\mathbb{N}$ by declaring $(x,y) \sim (a,b)$ if and only if $xb = ya$. Prove the following:

(a) $\sim$ is an equivalence relation

Proof. This was done above. □

(b) If we define $[(a,b)]+[(c,d)]$ to be $[(ad+bc,bd)]$ then addition on $\mathbb{Z}\times\mathbb{N}/\sim$ is well-defined.

Proof. Suppose that $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. We show that $[(a,b)]+[(c,d)] = [(a',b')]+[(c',d')]$. By definition of equivalence class, $(a,b) \sim (a',b')$ and $(c,d) = (c',d')$. By definition of the equivalence relation:

$$\begin{aligned} ab' &= a'b \\ cd' &= c'd \end{aligned}$$

We want to show that $[(ad+bc,bd)] = [(a'd'+b'c',b'd')]$. This is equivalent to showing:

$$(ad+bc,bd) \sim (a'd'+b'c',b'd')$$

To show that, by the definition of $\sim$ it suffices to show:

$$(ad+bc)b'd' = (a'd'+b'c')bd.$$

We now do that, using the two equations above.

$$\begin{aligned} (ad+bc)b'd' &= adb'd'+bcb'd' \\ &= (ab')dd'+(cd')bb' \\ &= a'bdd'+c'dbb' \\ &= (a'd'+b'c')bd \end{aligned}$$

which is what we want. □

(c) If we define $[(a,b)] \cdot [(c,d)]$ to be $[(ac,bd)]$ then multiplication on $\mathbb{Z} \times \mathbb{N}/\sim$ is well-defined.

Proof. Suppose that $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. We show that $[(a,b)] \cdot [(c,d)] = [(a',b')] \cdot [(c',d')]$. By definition of equivalence class, $(a,b) \sim (a',b')$ and $(c,d) = (c',d')$. By definition of the equivalence relation:

$$\begin{aligned} ab' &= a'b \\ cd' &= c'd \end{aligned}$$

Multiplying the first equation by cd':

$$ab'cd' = a'bcd'$$

Substitute on the right hand side using the second equation:

$$ab'cd' = a'bc'd$$

Thus,

$$(ac)(b'd') = (bd)(a'c').$$

By the definition of $\sim$:

$$(ac,bd) \sim (a'c',b'd').$$

Hence,

$$[(a,b)] \cdot [(c,d)] = [(ac,bd)] = [(a'c',b'd')] = [(a',b')] \cdot [(c',d')]$$

by the definition of multiplication. $\square$

(12) If $T\colon \mathbb{R} \to \mathbb{R}$ is a function such that $T(0) = 0$, $T(1) = 1$, and $|T(x) - T(y)| = |x - y|$ for all $x, y \in \mathbb{R}$, then $T = \mathrm{id}|_{\mathbb{R}}$.

Proof. Let T be as in the hypothesis of the statement. Note that by definition the function $\mathrm{id}|_{\mathbb{R}}$ has $\mathbb{R}$ as its domain and codomain, so T and $\mathrm{id}|_{\mathbb{R}}$ have the same domain and codomain. Let $x \in \mathbb{R}$ be arbitrary. We show that $T(x) = x$. Since $\mathrm{id}_{\mathbb{R}}(x) = x$, by definition, this will show that $T = \mathrm{id}|_{\mathbb{R}}$.

Since T is distance-preserving: $|T(x) - T(0)| = |x - 0| = |x|$. Since $T(0) = 0$, this means $|T(x)| = |x|$. Thus, $T(x) = \pm x$. If $T(x) = x$, we are done, so suppose that $T(x) = -x$. Likewise,

$$|-x-1| = |T(x) - 1| = |T(x) - T(1)| = |x - 1|$$

Thus, $-x - 1 = \pm(x - 1)$. If we use the minus sign on the right, then $-1 = +1$, a contradiction. Thus, we must use the plus sign. That is, $-x - 1 = x - 1$. Thus, $x = 0$. Since $T(0) = 0$ by assumption, we have $T(x) = x$, even in this case. Thus, $T(x) = x$, no matter what $x \in \mathbb{R}$ is. $\square$

(13) Let X be a nonempty set and let $\mathscr{D}$ be the set of all metrics on X. For $d, d' \in \mathscr{D}$ define $d \sim d'$ if and only if there exists $K \geq 1$, and $C \geq 0$ such that

$$\frac{1}{K} d(x,y) - C \leq d'(x,y) \leq K d(x,y) + C$$

for all $x, y \in X$. Prove that $\sim$ is an equivalence relation.

Proof. We show that $\sim$ is reflexive, symmetric, and transitive. For simplicity, in what follows we use function notation and write d rather than constantly writing things like $d(x,y)$ for all $x, y \in X$.

Let $d \in \mathscr{D}$. Choosing $K = 1$ and $C = 0$, we see that $d \sim d$.

Suppose $d, d' \in \mathscr{D}$ and $d \sim d'$. By definition of $\sim$, there exist $K \geq 1$ and $C \geq 0$ with

$$\frac{1}{K}d - C \leq d' \leq Kd + C$$

That is,

$$\begin{array}{rcl} d' & \leq & Kd + C \\ d' & \geq & \frac{1}{K}d - C \end{array}$$

Solving the two inequalities for d produces

$$\begin{array}{rcl} \frac{1}{K}d' - \frac{C}{K} & \leq & d \\ Kd' + CK & \geq & d \end{array}$$

Observe that $-CK \leq -\frac{C}{K}$, since $C \geq 0$and $K \geq 1$. Thus, we also have

$$\frac{1}{K}d' - CK \leq d.$$

Thus,

$$\frac{1}{K}d' - CK \leq d \leq Kd' + CK$$

Setting $C' = CK$, we have shown that there exist $C' \geq 0$ and $K \geq 1$ so that

$$\frac{1}{K}d' - C' \leq d \leq Kd' + C'.$$

Thus, $d' \sim d$.

Now suppose that $d_1, d_2, d_3 \in \mathscr{D}$ so that $d_1 \sim d_2$ and $d_2 \sim d_3$. Then by definition there exist $J, K \geq 1$ and $C, D \geq 0$ so that

$$\begin{array}{rcl} d_2 & \leq & Jd_1 + C \\ d_3 & \leq & Kd_2 + D \\ d_2 & \geq & \frac{1}{J}d_1 - C \\ d_3 & \geq & \frac{1}{K}d_2 - D \end{array}$$

Substituting the first inequality into the second and the third into the fourth (using the fact that $K \geq 0$ and $d_1 \geq 0$) we obtain:

$$\begin{array}{rcl} d_3 & \leq & KJd_1 + KC + D \\ d_3 & \geq & \frac{1}{KJ}d_2 - \frac{C}{K} - D. \end{array}$$

Let $K' = KJ$ and observe that $K' \geq 1$ since $K, J \geq 1$. Let $C' = KC + D$. Notice that since $K \geq 1$, we have $C' \geq \frac{C}{K} - D$. Thus,

$$\frac{1}{K'}d_1 - C' \leq d_3 \leq K'd_1 + C'.$$

Thus, $d_1 \sim d_3$, as desired. So $\sim$ is also transitive. $\square$

(14) (CHALLENGE) Consider the equivalence relation $\sim$ on $\mathscr{D}$ in the previous problem. Recall that if d and d' are metrics on X, then we can define $d + d'$ by

$$(d + d')(x, y) = d(x, y) + d'(x, y)$$

for all $x, y \in X$ and that $d + d'$ is a metric on X. Define $+$ on $\mathscr{D}/\sim$ by

$$[d] + [d'] = [d + d']$$

for all $d, d' \in \mathscr{D}$. Prove that $+$ is well-defined on $\mathscr{D}/\sim$.

(15) Let X be a set and let $\mathscr{F} = \{f\colon X \to \mathbb{R}\}$ be the set of all functions from X to $\mathbb{R}$. Recall that for $f, g \in \mathscr{F}$, the functions $f+g$ and $f \cdot g$ are defined by letting $(f+g)(x) = f(x)+g(x)$ and $f \cdot g(x) = f(x)g(x)$ for all $x \in X$. Define $\sim$ on $\mathscr{F}$ by declaring $f \sim f'$ if and only if there exists $M \geq 0$ such that $|f(x) - f'(x)| \leq M$ for all $x \in X$.

(a) Prove that $\sim$ is an equivalence relation on $\mathscr{F}$.

(b) Define $+$ and $\cdot$ on $\mathscr{F}/\sim$ by $[f]+[g] = [f+g]$ and $[f]\cdot[g] = [f \cdot g]$. Prove that $+$ is well-defined and $\cdot$ is not.

Proof. We show that $\sim$ is reflexive, symmetric, and transitive. It is reflexive, because $|f(x) - f(x)| = 0 \leq 1$, for all $x \in X$. Thus, $f \sim f$. It is reflexive, because if $f \sim g$, then for all $x \in X$, $|f(x)-g(x)| \leq M$. Thus, $|g(x)-f(x)| = |f(x)-g(x)| \leq M$ for all $x \in X$. Thus, $g \sim f$. Suppose that $f \sim g$ and $g \sim h$. By definition of $\sim$, there exist $M_1, M_2 \geq 0$ such that

$$\begin{array}{lcl} |f(x)-g(x)| & \leq & M_1 \text{ and} \\ |g(x)-h(x)| & \leq M_2 & \end{array}$$

for all $x \in \mathbb{R}$. Then adding our two equations:

$$M_1 + M_2 \geq |f(x)-g(x)| + |g(x)-h(x)| \geq |f(x)-g(x)+g(x)-h(x)| = |f(x)-h(x)|$$

for all $x \in \mathbb{R}$. (The second inequality comes from the triangle inequality for absolute value.) Thus, $f \sim h$ as desired.

Suppose that $[f] = [f']$ and $[g] = [g']$. We show $[f]+[g] = [f']+[g']$. Since $[f] = [f']$, $f \sim f'$. Thus, there exists $M_1 \geq 0$ such that $|f(x) - f'(x)| \leq M_1$ for every $x \in \mathbb{R}$. Similarly, there exists $M_2 \geq 0$ such that $|g(x) - g'(x)| \leq M_2$. Using the same trick as in our proof of transitivity, we add the inequalities:

$$M_1 + M_2 \geq |f(x)-f'(x)| + |g(x)-g'(x)| \geq |f(x)-f'(x)+g(x)-g'(x)| = |(f(x)+g(x)) - (f'(x)-g'(x))|$$

for all $x \in \mathbb{R}$. Thus, $(f+g) \sim (f'+g')$. Consequently, $[f+g] = [f'+g']$ and so $[f]+[g] = [f']+[g']$ as in the definition of addition for equivalence classes.

To see that multiplication is not well-defined, for all $x \in \mathbb{R}$, let $f(x) = e^x$ and $f'(x) = e^x + 5$. Let $g(x) = x$ and $g'(x) = x+3$. Note that $f \sim f'$ and $g \sim g'$. However, for all $x \in \mathbb{R}$:

$$\begin{array}{rcl} |f(x)g(x) - f'(x)g'(x)| & = & |xe^x - (x+3)(e^x+5)| \\ & = & |xe^x - xe^x - 3e^x - 5x - 15| \\ & = & |3e^x + 5x + 15| \end{array}$$

Since $\lim\limits_{x\to\infty} 3e^x + 5x + 15 = \infty$, the difference of the products is not bounded. Thus, $fg \not\sim f'g'$ and so multiplication is not well-defined on equivalence classes.

□

(16) Suppose that $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$ are functions. Prove

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(17) There is a bijection from the interval $(-10, 10)$ to the interval $(1, 2)$.

Proof. Let $f(x) = \frac{1}{20}(x-1)+2$ for all $x \in (-10, 10)$. Observe that $f\colon (-10,10) \to (1,2)$ is a function with inverse

$$f^{-1}(y) = 20(y-2)+1$$

for all $y \in (1,2)$. Since f has an inverse function it is a bijection. □

(18) Suppose that G is a group with operation $\circ$ and that H is a subgroup. For $a, b \in G$ define $a \sim b$ if and only if $a \circ b^{-1} \in H$. Prove that $\sim$ is an equivalence relation on G.

Proof. The axioms of a group would be given to you for this problem. Let G be a group and $H \subset G$ a subgroup. Define $a \sim b$ if and only if $a \circ b^{-1} \in H$. We show $\sim$ is reflexive, symmetric, and transitive.

Since $a \circ a^{-1} = \nVdash$ and since $\nVdash \in H$ as it is a group, $a \sim a$. Thus $\sim$ is reflexive.

Suppose that $a \sim b$. By definition $a \circ b^{-1} \in H$. Since H is a subgroup $(a \circ b^{-1})^{-1} \in H$. But $(a \circ b^{-1})^{-1} = b \circ a^{-1}$ as

$$(a \circ b^{-1}) \circ (b \circ a^{-1}) = a \circ (b \circ b^{-1}) \circ a^{-1} = \nVdash$$

and

$$(b \circ a^{-1}) \circ (a \circ b^{-1}) = b \circ (a^{-1} \circ a) \circ b^{-1} = \nVdash$$

by associativity and the definition of inverses. Thus, $b \circ a^{-1} \in H$, so $b \sim a$. Thus, $\sim$ is symmetric.

Finally, suppose that $a \sim b$ and $b \sim c$. Then $a \circ b^{-1} \in H$ and $b \circ c^{-1} \in H$. Since H satisfies the closure axiom

$$(a \circ b^{-1}) \circ (b \circ c^{-1}) \in H$$

However,

$$(a \circ b^{-1}) \circ (b \circ c^{-1}) = a \circ c^{-1}$$

by associativity, the definition of inverse, and the properties of the identity element. Thus, $a \circ c^{-1} \in H$ so $a \sim c$. Thus, $\sim$ is transitive. □

(19) Suppose that $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$ are functions. Prove

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof. Recall that by the definition of function composition, $g \circ f\colon A \to C$ and $h \circ g\colon B \to D$. Thus,

$$h \circ (g \circ f)\colon A \to D$$

and

$$(h \circ g) \circ f\colon A \to D$$

by the definition of function composition. Consequently, $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have the same domain and codomain.

Now for any $x \in A$, we have:

$$h \circ (g \circ f)(x) = h(g \circ f(x)) = h(g(f(x)))$$

and

$$(h \circ g) \circ f(x) = h \circ g(f(x)) = h(g(f(x))).$$

Thus,

$$h \circ (g \circ f)(x) = (h \circ g) \circ f(x)$$

for every $x \in A$. Thus, as $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have the same domain and codomain and produce the same output for an input, they are equal functions. □

(20) Use induction to prove that for all $n \in \mathbb{N}^*$ there exists $m \in \mathbb{N}^*$ such that $n = 3m$ or $n = 3m+1$ or $n = 3m+2$.

Proof. We induct on n. The base case is $n = 0$ for which observe that $0 = 3 \cdot 0$, so the statement holds with $m = 0$. Now suppose that there exists a $k \in \mathbb{N}^*$ such that there is an $m \in \mathbb{N}^*$ with $k = 3m + r$ for some $r \in \{0, 1, 2\}$. If $r \neq 2$, then $r + 1 \in \{1, 2\}$ and we have $k + 1 = 3m + 1$ or $k + 1 = 3m + 2$, as desired. If $r = 2$, then $k + 1 = 3m + 2 + 1 = 3(m + 1)$ and the result again holds as $m + 1 \in \mathbb{N}^*$. Thus by induction the result holds for all $n \in \mathbb{N}^*$. □

(21) Use induction to prove that every convex polygon with n sides can be divided into $(n-2)$ triangles using only edges with endpoints on the corners of the polygon.

Proof. We induct on $n \geq 3$. When $n = 3$, the polygon is a triangle and it is automatically tiled with $(n-3) = 1$ triangles as required. Suppose that there is a $k \geq 3$ such that every convex polygon with k sides can be tiled with $k-2$ triangles using edges that have their endpoints on the corner of the polygon. Let P be a convex polygon with $k+1$ sides. There are corners v_1, v_2, v_3 such that $v_1 \neq v_3$ and v_1 and v_2 are the endpoints of a side of P as are v_2 and v_3. Let e be the line segment in $\mathbb{R}^2$ joining v_1 to v_3. Since $k+1 = 4$, this line segment divides P into two polygons A and B, with one of them (say B) being a triangle and the other (A) having k sides (one of which is e). By our induction hypothesis, A can be tiled with $(k-2)$ triangles, each with their edges on the corners of A. Since B is a triangle, those triangles together with B tile P and the edges of the triangles have their endpoints on the corners of P. By induction, every polygon with $n \geq 3$ sides can be subdivided into $(n-2)$ triangles using only edges with endpoints on the corners. □

(22) Let $x_0 = \sqrt{2}$ and define $x_{n+1} = \sqrt{2+x_n}$ for all $n \in \mathbb{N}^*$. Prove that for all $n \in \mathbb{N}^*$, $x_{n+1} > x_n$.

Proof. We induct on n. For $n = 0$, $x_1 = \sqrt{2+\sqrt{2}} \geq \sqrt{2} = x_0$. Suppose there is a $k \in \mathbb{N}^*$ such that $x_{k+1} > x_k$. Then observe that:

$$x^2_{(k+1)+1} = (\sqrt{2+x_{k+1}})^2 = 2 + x_{k+1} \geq 2 + x_k$$

where the inequality arises from the inductive hypothesis. Taking the square root of both sides:

$$x_{(k+1)+1} \geq \sqrt{2+x_k} = x_{k+1}.$$

Thus, by induction, $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}^*$.

□

(23) Give a thorough outline of a proof that if f is a permutation of a set X with $n \geq 2$ elements, then f is the composition of transpositions. (A transposition is a permutation such that there exists distinct a, b such that $f(a) = b$ and $f(b) = f(a)$ and $f(x) = x$ for all $x \neq a, b$.)

Proof. We induct on the number n of elements of X. If $n = 2$, then $X = \{a, b\}$ with $a \neq b$. A bijection $f\colon \{a, b\}$ is either a transposition or the identity. The result holds automatically if f is a transposition. If it is the identity, observe it is the composition of the transposition interchanging a and b with itself.

Suppose that for some $k \geq 2$, every permutation of a set with k elements is the composition of permutations. Let $f\colon X \to X$ be a permutation of a set X having $k+1$ elements. Let $x_0 \in X$. If $f(x_0) = x_0$, then $g\colon X \setminus \{x_0\} \to X \setminus \{x_0\}$ defined by $g(x_0) = x_0$ is the composition of transpositions. Extending those transpositions to have domain and codomain equal to X, we see that f is also a composition of transpositions.

Suppose, therefore, that $f(x_0) \neq x_0$. Let $\tau\colon X \to X$ be the transposition such that $\tau(x_0) = f(x_0)$ and $\tau(f(x_0)) = x_0$. Then $\tau \circ f(x_0) = x_0$. Consequently, by the previous paragraph, there exist transpositions $\tau_1, \ldots, \tau_m$ such that

$$\tau \circ f = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m.$$

Noting that $\tau \circ \tau$ is the identity, we have:

$$f = \tau \circ \tau \circ f = \tau \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m.$$

Thus, f is also a composition of transpositions. $\square$

(24) (CHALLENGE) Let X be the set of all real-valued functions on the vertices V of a graph G having directed edges and let Y be the set of all real valued functions on the edges E of G. If e is a directed edge of G, let e_- be the tail of e and e_+ be the head of e. Define $\nabla\colon X \to Y$ by declaring $\nabla(f)\colon E \to \mathbb{R}$ to be the function defined by $\nabla(f)(e) = f(e_+) - f(e_-)$ for all $e \in E$. Prove that ∇ is surjective if and only if G has no cycles.

Proof. (sketch)

Suppose first that ∇ is surjective. We prove that G has no cycles by contradiction. Suppose that $v_0, v_1, v_2, \dots, v_n$ are the vertices of a cycle ($n \geq 1$). That is for each $i \in \{0, \dots, n-1\}$ there is an edge E_i pointing from v_i to v_{i+1} and there is also an edge E_n pointing from v_n to v_0. Let $g \in Y$ be the function such that $g(E_0) = 1$ and $g(e) = 0$ for every edge of G other than E_0. Suppose that there exists $f \in X$ such that $\nabla f = g$. Suppose also that $f(v_0) = c \in \mathbb{R}$. Then $1 = \nabla g(E_0) = f(v_1) - f(v_0)$. Thus, $f(v_1) = 1 + c$. By induction, $f(v_n) = n + c$. However, then

$$0 = \nabla g(E_n) = f(v_0) - f(v_n) = c - (n + c) = -n.$$

This contradicts the fact that $n \geq 1$.

Suppose now that G has no cycles. We prove that ∇ is surjective. Let $g \in Y$. Without loss of generality, assume G is connected (otherwise, repeat the following argument in each component.) Let v_0 be a vertex of G. Since G has no cycles, for each vertex $v \in G$, there is a unique path from v_0 to v. Let $e_0, e_1, \dots, e_m$ be the edges of that path. (The path may travel backwards along some edges.) Let $v_1, v_2, \dots, v_{m+1} = v$ be the vertices of the path so that v_i and v_{i+1} are the endpoints of e_i. Let $f(v) = \pm g(e_0) \pm g(e_1) \pm \cdots \pm g(e_m)$, where each sign is determined by whether the path travels in the same direction as e_i (in which case, use +) or in the opposite direction (in which case use $-$). For an edge e in the path, the value of $\nabla(f)$ is the difference of the values of f on the endpoints. It follows that it is equal to g. (more details needed) $\square$

(25) (Bonus solution) There is a bijection from the interval $[0,1]$ to the interval $(0,1)$.

Proof. For all $n \in \mathbb{N}$, let $s_n = \frac{1}{2} + \frac{1}{2n}$. Observe that if $s_n = s_m$ then $n = m$. For all $n \in \mathbb{N}$, let $t_n = \frac{1}{2} - \frac{1}{2n}$. Again notice that if $t_n = t_m$ then $n = m$. Also, there does not exist $n, m \in \mathbb{N}$ such that $s_n = t_m$. Consequently, we have a function $f\colon [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} s_{n+1} & \text{if there exists } n \in \mathbb{N} \text{ with } x = s_n \\ t_{m+1} & \text{if there exists } m \in \mathbb{N} \text{ with } x = t_m \\ x & \text{otherwise} \end{cases}$$

Furthermore, since $s_1 = 1$ and $t_1 = 0$, neither 0 or 1 are in the range of f and that range $f = (0,1)$. Restricting the codomain of f to $(0,1)$, we have a function $f\colon [0,1] \to (0,1)$. It has inverse $f^{-1}\colon (0,1) \to [0,1]$ given by

$$f^{-1}(x) = \begin{cases} s_{n-1} & \text{if there exists } n \in \mathbb{N} \text{ with } x = s_n \text{ and } n \geq 2 \\ t_{m-1} & \text{if there exists } m \in \mathbb{N} \text{ with } x = t_m \text{ and } m \geq 2 \\ x & \text{otherwise .} \end{cases}$$

Thus, since f has an inverse it is a bijection. $\square$

(26) (Bonus solution) Suppose that $a, b \in \mathbb{N}$. Show there exist $q, r \in \mathbb{N}^*$ such that $b = aq + r$ and $r < 0$.

Proof. We induct on b, so consider $a \in \mathbb{N}$ to be fixed.

Base Case: $b = 1$.

If $a = 1$, then $b = a(b) + 0$, so letting $q = b$ and $r = 0$, we have the desired result. If $a > 1$, then $b = a(0) + 1$ and letting $q = 0$ and $r = 1 < a$ concludes the base case.

Inductive Step: Assume that there is a $k \in \mathbb{N}$ such that there are $q, r \in \mathbb{N}^*$ with $k = aq + r$ and $r < k$. We prove that there exist $q', r' \in \mathbb{N}^*$ with $k + 1 = aq' + r'$ and $r' < a$. We split the proof into two cases: $r < a - 1$ and $r = a - 1$.

Case 1: $r < a - 1$.

Let $q' = q$ and $r' = r + 1$. Then $k + 1 = (aq + r) + 1 = aq + (r + 1) = aq' + r'$. Since $r < a - 1$, $r' < a$.

Case 2: $r = a - 1$.

Let $q' = q + 1$ and $r' = 0$. For then $k + 1 = (aq + r) + 1 = aq + (a - 1) + 1 = a(q + 1) = aq' + r'$. Since $r' < a$, we have the desired result.

$\square$