## Fall 2018 Exam 1

You may not use textbooks, notes, or refer to other people (except the instructor). You may not have any internet-connected or connectable devices visible to you or others. If you find a problem statement ambiguous, feel free to ask for clarification. Remember to make the structures of your proofs clear.
Please do not place any answers on the exam itself. Instead turn in your final answers in the proper order, and with all preliminary work clearly labelled and attached at the very end of what you turn in. Turn in this cover sheet as the first page of your exam.
Place your name and problem number on every page. You do not have to recopy the problem statement.

| Problem | Score | Possible |
| :---: | :---: | :---: |
| 1 |  | 20 |
| 2 |  | 15 |
| 3 |  | 10 |
| 4 |  | 10 |
| 5 |  | 10 |
| 6 |  | 10 |
| 7 |  | 15 |
| 8 |  | 10 |
| 9 |  | 5 (extra-credit) |
| Total |  | 100 |

Name (printed): $\qquad$

## Definitions

A graph consists of a set of vertices $V$ and a set of edges $E$. For each edge $e \in E$, there exist $v, w \in V$ such that $v$ and $w$ are the endpoints of $e$.

A path in a graph $G$ from vertex $x$ to vertex $y$ is a list

$$
v_{0}, v_{1}, \ldots, v_{n}
$$

where $n \in \mathbb{N}^{*}$ and for each $i \in\{0, \ldots, n\}, v_{i}$ is a vertex of $G$. Furthermore, for all $i \in\{0, \ldots, n-1\}$, the vertices $v_{i}$ and $v_{i+1}$ are endpoints of an edge in $G$.
(1) (20 pts) Give a precise definition of the following terms. You do not need to define the terms in parentheses.
(a) statement
(b) universal quantifier
(c) the union of sets $A_{\lambda}$ (where each $\lambda$ is an element of some index set $\Lambda$ ).
(d) The power set of a set $X$.
(2) ( 15 pts ) Write the negations of the following statements. Phrase your answer as positively as possible. Statements in parentheses are not part of what you are to negate.
(a) There exists $a \in \mathbb{R}^{2}$ such that for all $b \in \mathbb{R}^{2}, d(a, b) \leq 4$.
(b) If $a^{2}+b^{2} \leq 1$, then $a=0$ or $b=0$.
(c) ( $A$ is some fixed subset of $\mathbb{R}$ and $M \in \mathbb{R}$.) For every $a \in A, a \leq M$ and for every $\epsilon>0$, there exists $x \in A$ such that $M-\epsilon<x$.
(3) (10 pts) Prove that there exist infinitely many prime numbers. If you need other facts, say what they are and whether or not you know how to prove them.
(4) ( 10 pts ) Suppose that $A_{\lambda}$ is a set for all $\lambda \in \Lambda$ (where $\Lambda$ is some index set.) Prove that

$$
\bigcup_{\lambda \in \Lambda}\left(A_{\lambda}^{C}\right) \subset\left(\bigcap_{\lambda \in \Lambda} A_{\lambda}\right)^{C} .
$$

(5) (10 pts) Refer to the 2nd page of the exam for needed definitions. Prove that if $a, b$, and $c$ are vertices of a graph such that there is a path from $a$ to $b$ and a path from $b$ to $c$, then there is also a path from $a$ to $c$. Your solution needs to explicitly reference the exact definitions.
(6) (10 pts) Let $X=\left\{(x, y) \in \mathbb{R}^{2}: y=x^{2}\right\}$. Let $Y=\left\{(x, y) \in \mathbb{R}^{2}: y=(x-1)^{2}\right\}$. Prove that there is a unique element of $X \cap Y$.
(7) (15 pts) Suppose that $X$ and $Y$ are non-empty sets. Prove that $X \times Y=Y \times X$ if and only if $X=Y$.
(8) (10 pts) Suppose that $X$ is a set and that $\mathscr{T} \subset \mathscr{P}(X)$. Then $\mathscr{T}$ is a topology on $X$ if the following hold:
(T1) $\varnothing \in \mathscr{T}$ and $X \in \mathscr{T}$
(T2) If $A \in \mathscr{T}$ and $B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$.
(T3) If $U_{\lambda} \in \mathscr{T}$ for all $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} U_{\lambda} \in \mathscr{T}$
Suppose that $\mathbb{T}$ is a non-empty set such that every element $\mathscr{T} \in \mathbb{T}$ is a topology on $X$. Prove that

$$
\bigcap_{\mathscr{T} \in \mathbb{I}} \mathscr{T}
$$

is a topology on $X$.
(9) (5 pts Extra-Credit) Explain what Russell's paradox is and what its significance is.

