F22 MA 274: Exam 3 Study Questions

These questions mostly pertain to material covered since Exam 2, but there are some from just before Exam 2. New problems are marked. The final exam is cumulative, so you should also study earlier material, including the earlier study guides. Some solutions will be posted at a later date - so get started now!

(1) Know the definitions on the website. Any other definitions that you need will be given to you.

(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(3) Know the theorems we've proved in class and the more significant theorems from the homework.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(6) Study the previous study guides and exams as well as your homework, class notes, and the sections of the text we covered.

Prove the following:

(1) Suppose that (x_n) is a sequence in a set X such that for all $N \in \mathbb{N}$ there exists $m \geq N$ with $x_m \not\in \{x_1, \dots, x_N\}$. Prove that there exists an *injective* sequence (a_k) in X such that $\mathrm{range}(a_k) = \mathrm{range}(x_n)$. BONUS: Show that we may define the sequence (a_k) in such a way that there is a strictly increasing sequence (n_i) in $\mathbb{N}$ with $a_k = x_{n_k}$ for all $k \in \mathbb{N}$.

(2) (NEW) Suppose that (x_n) is a sequence in $\mathbb{R}$ with the property that for all $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_m < \min\{x_1, \dots, x_N\}$. Recursively define a sequence (n_k) in $\mathbb{N}$ such that $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ and $x_{n_{k+1}} < x_{n_k}$ for all $k \in \mathbb{N}$.

(3) If P is a (convex) polygon with $n \geq 3$ sides, then P has a triangulation with $n - 2$ triangles and all vertices of the triangulation are also vertices of P.

(4) Prove that if G is a finite, connected, non-empty planar graph, then the number of vertices minus the number of edges plus the number of faces equals 2.

(5) (NEW) Prove that if T is a tree (that is, a connected graph without cycles) with at least one edge, then T has at least two leaves (i.e. vertices that are each incident to a single edge of T).

(6) Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a permutation such that there exists $N \in \mathbb{N}$ with $f(n) = n$ for all $n > N$. Prove that there exist transpositions $\tau_1, \cdot, \tau_k$ such that

$$f = \tau_1 \circ \cdots \circ \tau_k$$

(7) Prove that for every natural number $n \geq 2$, there exist prime numbers $p_1, p_2, \dots, p_k$ such that $n = p_1 p_2 \cdots p_k$.

(8) (NEW) Prove that for every rational number $r \in \mathbb{Q}$, there exist $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $r = a/b$ and a and b have no common factor.

(9) Prove that if a and b are natural numbers, then there exist $q, r \in \mathbb{N}^*$ such that $b = aq + r$ and $r < a$.

(10) (NEW) Prove that if e is an edge of a connected graph G and if $G - e$ results from removing the edge e (but leaving its endpoints) then $G - e$ has at most two components.

(11) Prove that a connected nonempty graph where every vertex has even degree has an Euler circuit.

(12) (NEW) Prove that if α is a path from a vertex a to a different vertex b in a graph G, then either α does not pass through any vertex twice or there is a path from a to b which contains fewer vertices than α.

(13) If X is a set such that there is an injection $f\colon X \to B$ where B is a proper subset of X, then X is infinite.

(14) Prove that if n is a natural number, then there exists $m \in \mathbb{N}$ and digits $d_i \in \{0, 1, 2\}$ for $i \in \{0, \dots, m\}$ such that

$$n = \sum_{i=0}^{m} d_i 3^i$$

(In other words, natural numbers can be written in ternary notation.)

(15) If X is an infinite set, then there exists an infinite injective sequence in X

(16) Prove that a subset of a countable set is countable, for instance by showing that if X has a bijective sequence and if $A \subset X$ is not finite, then A also has a bijective sequence.

(17) Prove that the following sets are countable:

(a) $\mathbb{Z}$
(b) $\mathbb{N} \times \mathbb{N}$
(c) $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$
(d) $\bigcup_{\lambda \in \Lambda} A_\lambda$ where Λ is a non-empty countable set and each A_λ is non-empty and countable.
(e) (NEW) $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k}$

(18) Prove that for every set X, $\operatorname{card} X < \operatorname{card} \mathcal{P}(X)$.

(19) The set of sequences in $\{0, 1\}$ is uncountable.

(20) The interval $[0, 1)$ is uncountable.

(21) If X has an infinite injective sequence in X, then for any element $a \in X$, $\operatorname{card} X = \operatorname{card} X \setminus \{a\}$.

(22) If there exists $a \in X$ such that $\operatorname{card} X = \operatorname{card} X \setminus \{a\}$, then X has an infinite injective sequence.

(23) If X and Y are sets such that there is an injection $f\colon X \to Y$, then there exists a surjection $g\colon Y \to X$.

(24) Let S^1 be the unit circle. For any $\alpha \in \mathbb{R}$, let R_α be the counterclockwise rotation by α radians. (If $\alpha < 0$ this means rotate by $|\alpha|$ radians clockwise.) Suppose that $\theta \in \mathbb{R}$. Let (x_n) be the sequence in S^1 where $x_0 = (1, 0)$ and $x_n = R_\theta(x_{n-1})$ for all $n \in \mathbb{N}$. Prove the following:

(a) The sequence (x_n) is injective if and only if $\theta \notin \pi\mathbb{Q}$ (i.e. θ is not a rational multiple of π.)

(b) The sequence (x_n) is periodic (i.e. there exists $n \in \mathbb{N}$ such that $x_n = x_0$) if and only if θ is a rational multiple of π.
(c) The sequence (x_n) is not surjective.

(25) Prove that the set of algebraic numbers is countable and, therefore, that the set of transcendental numbers is uncountable.

(26) (NEW) Let $X = \mathcal{P}(\mathbb{R})$ and define $\sim$ on X by $A \sim B$ if and only if there exists a bijection $f\colon A \to B$. Prove that $\sim$ is an equivalence relation.

(27) Let X be a non-empty set and let $\mathcal{F}$ be the set of bijections of X to itself (i.e. permutations of X). For $f, g \in \mathcal{F}$ define $f \sim g$ if and only if there exists a bijection $h \in \mathcal{F}$ such that

$$f = h^{-1} \circ g \circ h.$$

Prove that $\sim$ is an equivalence relation.

(28) (NEW) Let G be a group and $H \subset G$ a subgroup. Define $\sim_H$ on G by declaring $x \sim_H y$ if and only if there exists $h \in H$ with $x = h \circ y$.

(a) Prove that $\sim_H$ is an equivalence relation.
(b) Let $a \in G$ and let $[a]$ be its equivalence class under h. Define $f\colon H \to [a]$ by $f(h) = h \circ a$. Prove that f is a bijection.
(c) Explain why H and $[a]$ have the same cardinality.
(d) Conclude that if G is finite, then $|G/\sim_H| = |G|/|H|$. (This is Lagrange's Theorem in Group Theory)