## MA 274: Exam 2 Study Guide

Here are some suggestions for what and how to study:
Theorems marked (CHALLENGE) are particularly challenging and would placed on the exam only as a challenge problem.
(1) Know the definitions on the website up through Chapter 8. Any other definitions that you need will be given to you.
(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.
(3) Know the theorems we've proved in class and the more significant theorems from the homework.
(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.
(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.
(6) Here are some results you should be especially sure to know how to prove; some of them may be new. You should also think about ways these problems might be varied.
(7) Go back and look at the study guide for Exam 1 by way of reviewing the basics. The material from that exam will not be explicitly tested in Exam 2, but of course it forms the basis for what we do.

Here are some sample problems.
(1) Prove the following are examples of equivalence relations:

- $\equiv_{p}$ on $\mathbb{Z}$, where $x \equiv_{p} y$ iff $x-y$ is a multiple of $p$.
- $\sim$ on $\mathbb{Z} \times \mathbb{N}$ where $(a, b) \sim(c, d)$ iff $a d=b c$.
- $\sim$ on the vertex set of a graph $G$, where $x \sim y$ iff there is a path from $x$ to $y$ in the graph.
- Suppose that $P$ is a partition of a non-empty set $X$. Define $\sim_{P}$ on $X$ by $x \sim_{P} y$ iff there exists $A \in P$ such that $x, y$ are both elements of $A$.
(2) Suppose that $\sim$ is an equivalence relation on a non-empty set $X$. For each $x \in X$, let $[x]$ be the equivalence class of $x$. Then the following hold:
(a) For all $x \in X, x \in[x]$.
(b) For all $x, y \in X, x \sim y$ iff $[x]=[y]$.
(c) For all $x, y \in X,[x] \cap[y] \neq \varnothing$ implies $[x]=[y]$.
(3) If $\sim$ is an equivalence relation on a non-empty set $X$, then the set $X / \sim$ of equivalence classes is a partition of $X$.
(4) If $f: \mathbb{Z} / \equiv_{p} \rightarrow \mathbb{Z} / \equiv_{p}$ is defined by $f([x])=[2 x]$ then $f$ is well-defined.
(5) Addition and multiplication in $\mathbb{Z} / \equiv_{p}$ are well-defined. That is, define $[x]+[a]=[x+a]$ and $[x][a]=$ $[x a]$. Then if $[x]=\left[x^{\prime}\right]$ and $[a]=\left[a^{\prime}\right]$ then $[x]+[a]=\left[x^{\prime}\right]+\left[a^{\prime}\right]$ and $[x][a]=\left[x^{\prime}\right]\left[a^{\prime}\right]$.
(6) The compositions of injections/surjections/bijections is a an injection/surjection/bijection.
(7) A function $f: X \rightarrow Y$ is a bijection if and only if there is a function $f^{-1}: X \rightarrow Y$ such that $f \circ$ $f^{-1}(y)=y$ for all $y \in Y$ and $f^{-1} \circ f(x)=x$ for all $x \in X$.
(8) The set of bijections from a set $X$ to itself is a group, with function composition as the operation.
(9) The function $f: \mathbb{Z} / \equiv_{10} \rightarrow \mathbb{Z} / \equiv_{10}$ defined by $f([x])=[2 x]$ is not injective or surjective.
(10) The function $f: \mathbb{Z} / \equiv_{10} \rightarrow \mathbb{Z} / \equiv_{10}$ defined by $f([x])=[3 x]$ is injective and surjective.
(11) Define $\sim$ on $\mathbb{Z} \times \mathbb{N}$ by declaring $(x, y) \sim(a, b)$ if and only if $x b=y a$. Prove the following:
(a) $\sim$ is an equivalence relation
(b) If we define $[(a, b)]+[(c, d)]$ to be $[(a d+b c, b d)]$ then addition on $\mathbb{Z} \times \mathbb{N} / \sim$ is well-defined.
(c) If we define $[(a, b)] \cdot[(c, d)]$ to be $[(a c, b d)]$ then multiplication on $\mathbb{Z} \times \mathbb{N} / \sim$ is well-defined.
(12) If $T: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $T(0)=0, T(1)=1$, and $|T(x)-T(y)|=|x-y|$ for all $x, y \in \mathbb{R}$, then $T=\left.\mathrm{id}\right|_{\mathbb{R}}$.
(13) Let $X$ be a nonempty set and let $\mathscr{D}$ be the set of all metrics on $X$. For $d, d^{\prime} \in \mathscr{D}$ define $d \sim d^{\prime}$ if and only if there exists $K \geq 1$, and $C \geq 0$ such that

$$
\frac{1}{K} d(x, y)-C \leq d^{\prime}(x, y) \leq K d(x, y)+C
$$

for all $x, y \in X$. Prove that $\sim$ is an equivalence relation.
(14) (CHALLENGE) Consider the equivalence relation $\sim$ on $\mathscr{D}$ in the previous problem. Recall that if $d$ and $d^{\prime}$ are metrics on $X$, then we can define $d+d^{\prime}$ by

$$
\left(d+d^{\prime}\right)(x, y)=d(x, y)+d^{\prime}(x, y)
$$

for all $x, y \in X$ and that $d+d^{\prime}$ is a metric on $X$. Define + on $\mathscr{D} / \sim$ by

$$
[d]+\left[d^{\prime}\right]=\left[d+d^{\prime}\right]
$$

for all $d, d^{\prime} \in \mathscr{D}$. Prove that + is well-defined on $\mathscr{D} / \sim$.
(15) Let $X$ be a set and let $\mathscr{F}=\{f: X \rightarrow \mathbb{R}\}$ be the set of all functions from $X$ to $\mathbb{R}$. Recall that for $f, g \in \mathscr{F}$, the functions $f+g$ and $f \cdot g$ are defined by letting $(f+g)(x)=f(x)=g(x)$ and $f \cdot g(x)=f(x) g(x)$ for all $x \in X$. Define $\sim$ on $\mathscr{F}$ by declaring $f \sim f^{\prime}$ if and only if there exists $M \geq 0$ such that $\left|f(x)-f^{\prime}(x)\right| \leq M$ for all $x \in X$.
(a) Prove that $\sim$ is an equivalence relation on $\mathscr{F}$.
(b) Define + and $\cdot$ on $\mathscr{F} / \sim$ by $[f]+[g]=[f+g]$ and $[f] \cdot[g]=[f \cdot g]$. Prove that + is well-defined and $\cdot$ is not.
(16) Suppose that $G$ is a group with operation $\circ$ and that $H$ is a subgroup. For $a, b \in G$ define $a \sim b$ if and only if $a \circ b^{-1} \in H$. Prove that $\sim$ is an equivalence relation on $G$.
(17) Suppose that $f: A \rightarrow B, g: B \rightarrow C$, and $h: C \rightarrow D$ are functions. Prove

$$
h \circ(g \circ f)=(h \circ g) \circ f
$$

(18) Use induction to prove that for all $n \in \mathbb{N}^{*}$ there exists $m \in \mathbb{N}^{*}$ such that $n=3 m$ or $n=3 m+1$ or $n=3 m+2$.
(19) Use induction to prove that every convex polygon with $n$ sides can be divided into $(n-2)$ triangles using only edges with endpoints on the corners of the polygon.
(20) Let $x_{0}=\sqrt{2}$ and define $x_{n+1}=\sqrt{2+x_{n}}$ for all $n \in \mathbb{N}^{*}$. Prove that for all $n \in \mathbb{N}^{*}, x_{n+1}>x_{n}$.
(21) Give a thorough outline of a proof that if $f$ is a permutation of a set $X$ with $n \geq 2$ elements, then $f$ is the composition of transpositions. (A transposition is a permutation such that there exists distinct $a, b$ such that $f(a)=b$ and $f(b)=f(a)$ and $f(x)=x$ for all $x \neq a, b$.)
(22) (CHALLENGE) Let $X$ be the set of all real-valued functions on the vertices $V$ of a graph $G$ having directed edges and let $Y$ be the set of all real valued functions on the edges $E$ of $G$. If $e$ is a directed edge of $G$, let $e_{-}$be the tail of $e$ and $e_{+}$be the head of $e$. Define $\nabla: X \rightarrow Y$ by declaring $\nabla(f): E \rightarrow \mathbb{R}$ to be the function defined by $\nabla(f)(e)=f\left(e_{+}\right)-f\left(e_{-}\right)$for all $e \in E$. Prove that $\nabla$ is surjective if and only if $G$ has no cycles.

