F22 MA 274: Exam 1 Study Guide

Here are some suggestions for what and how to study:

(1) Know the definitions on the website from Chapters 1 - 5. Any other definitions/axioms that you need will be given to you. You will be asked for the precise statement of some of those definitions.

(2) Be able to write the negation of a mathematical statement, paying particular attention to how conjunctions, disjunctions, implications, and quantifiers behave under negation.

(3) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) When studying, practice writing proofs, don't just look at them.

(6) Know the basic kinds of proofs: existence proofs, uniqueness proofs, element arguments, direct proofs, proofs by contraposition, proofs by contradiction. What are examples of each?

(7) Study the theorems we've proved in class and the more significant theorems from the homework.

(8) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(9) There may be one or two theorems to prove on the exam that you haven't seen before. You should have a plan for how you will handle being confronted with a new problem.

(10) Here are some results you should study. You should also think about ways these problems might be varied. And you should study other problems too. On the exam axioms and definitions you weren't required to memorize will be given to you.

   (a) Suppose $n \in \mathbb{N}$. The number n is a multiple of 2 if and only if n^2 is a multiple of 2.

   (b) Suppose $n \in \mathbb{N}$. Then n and $n+1$ are both multiples of some $m \in \mathbb{N}$ if and only if $m = 1$.

   (c) The number $\sqrt{2}$ is irrational.

   (d) There are infinitely many prime numbers.

   (e) There is no set U such that $A \in U$ if and only if A is a set. (Russell's Paradox)

   (f) The Halting Problem

   (g) DeMorgan's Laws

   (h) Let S be the set whose elements are precisely the circles in $\mathbb{R}^2$ that are centered at the origin. Let T be the set whose elements are precisely the circles in $\mathbb{R}^2$ that intersect the line $x = 5$ at a single point. Prove that $S \cap T$ has a unique element.

   (i) Let a, b, c, d be real numbers such that $a < c < b < d$. Prove that $(a,b) \cup (c,d) = (a,d)$.

   (j) $A \cap \left(\bigcup_{\lambda \in \Lambda} B_\lambda \right) = \bigcup_{\lambda \in \Lambda} (A \cap B_\lambda)$

(k) Suppose G is a group with operation $\circ$ and that $a \in G$. If $f, g \in G$ have the properties that $f \circ a = a \circ f = a$ and $g \circ a = a \circ g = a$, then $f = g$. (That is, the identity in a group is unique.)

(l) Suppose that G is a group with operation $\circ$ and identity $\mathbf{1}$. Let $a \in G$. If $f, g \in G$ have the properties that $f \circ a = \mathbf{1}$ and $g \circ a = \mathbf{1}$, then $f = g$. (That is, inverses in groups are unique.)

(m) For integers $x, y \in \mathbb{Z}$, define $x \equiv y$ to mean that $x - y$ is a multiple of 17. Prove:

  (i) For all $x \in \mathbb{Z}$, $x \equiv x$.

  (ii) For all $x, y \in \mathbb{Z}$, if $x \equiv y$ then $y \equiv x$

  (iii) For all $x, y, z \in \mathbb{Z}$, if $x \equiv y$ and $y \equiv z$, then $x \equiv z$.

(n) The intersection of convex sets is convex

(o) The intersection of subgroups is a subgroup

(p) The intersection of event spaces is an event space.

(q) $X \times Y = Y \times X$ if and only if either $X = Y$ or one of X or Y is empty.

(r) If A and B are sets, then $A = B$ if and only if $\mathscr{P}(A) = \mathscr{P}(B)$.

(s) Remind yourself of the axioms of a metric space and suppose that X is a metric space with metric d. A subset $U \subset X$ is defined to be **open** if for every $a \in U$, there exists $r > 0$ such that the ball $B_r(a) = \{x \in X : d(x, a) < r\}$ is a subset of U. Suppose that $\mathscr{U}$ is a set such that if $U \in \mathscr{U}$ then U is an open set in X. Prove that the set $\bigcup_{U \in \mathscr{U}} U$ is open.