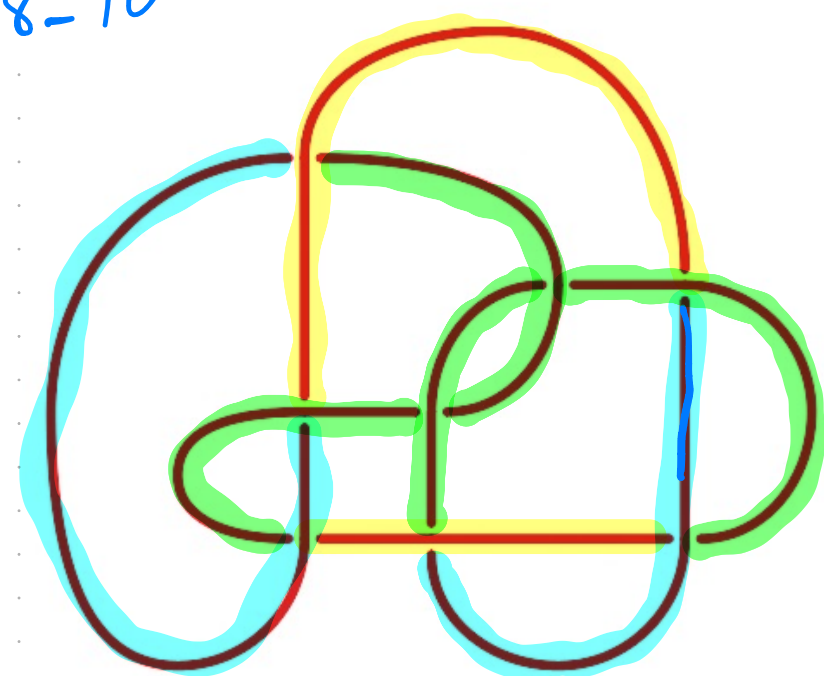
How can we tell if a knot is tricolorable?

8-10



Rules

- Can use 3 colors
- At each crossing use either 1 or 3 colors
- ~~Have to use all 3 colors in diagram~~

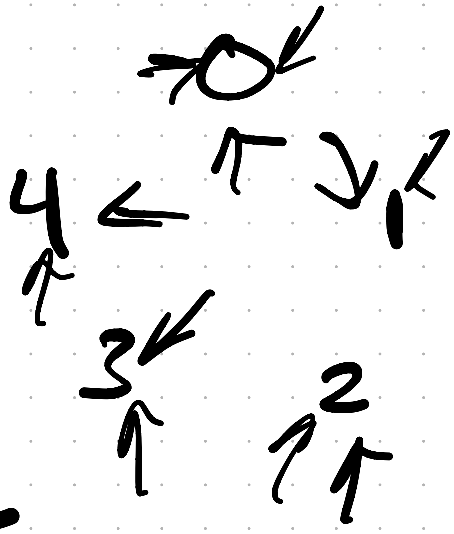
Def A coloring is non-trivial if we use more than 1 color.

K-colorability

Work w/ $\mathbb{Z}/k\mathbb{Z}$

Use modular arithmetic

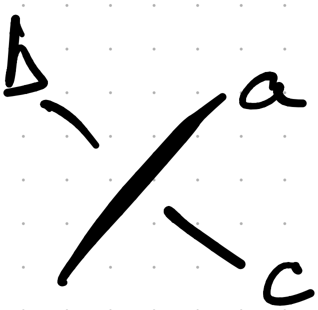
<u>Ex</u> $K = 5$



$1+2=3$

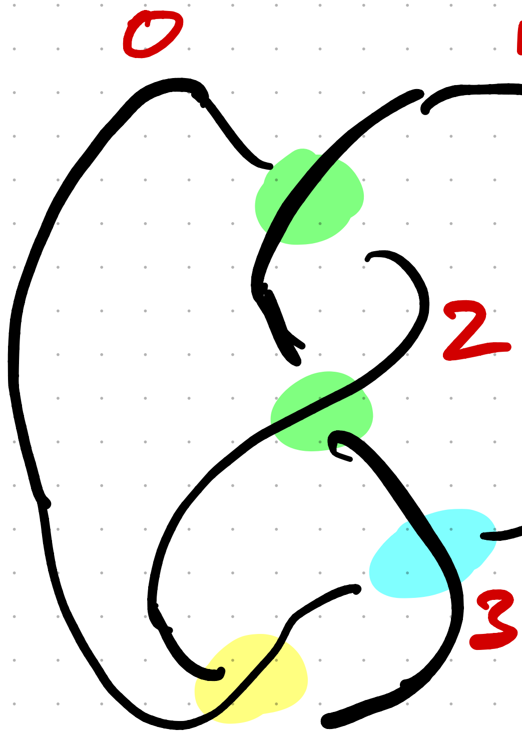
$2+3=0 \mod 5$

$2\cdot 3 = 6 = 1 \mod 5$



$2a - b - c = 0 \pmod{k}$

Fig 8 Knot is 5-colorable



$\{0, 1, 2, 3, 4\}$

$(k = 5)$

$2(1) - 0 - ? = 0 \bmod 5$

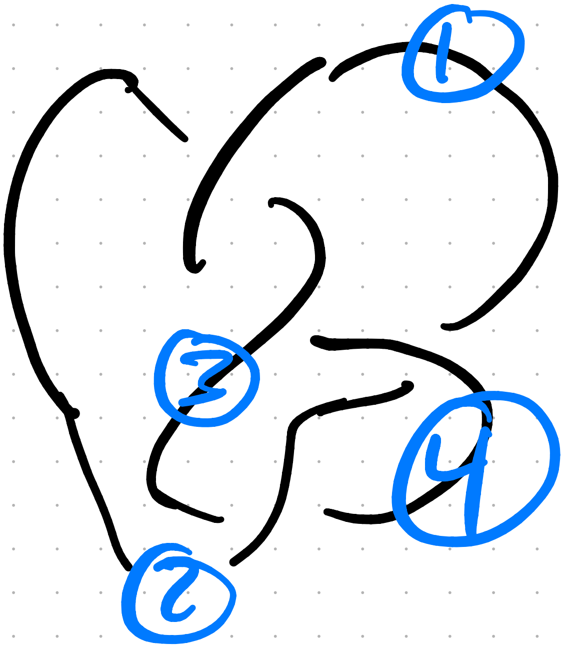
$2(2) - 1 - ? = 0 \bmod 5$

$3 - ? = 0 \bmod 5$

$2(3) - 0 - 1 \overset{?}{=} 0 \bmod 5$

$6 - 1 = 0 \bmod 5$ ✓

$2(0) - 2 - 3 = 0 \bmod 5$ ✓



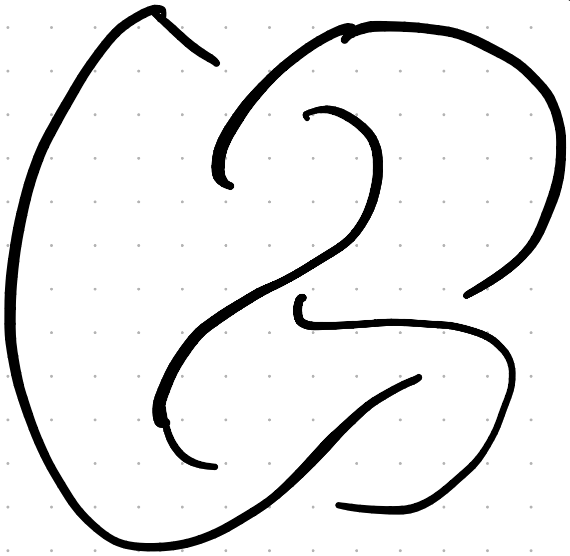
A coloring is a function from strands $\to \mathbb{Z}/k\mathbb{Z}$

A coloring is a <u>vector</u>

$$\begin{pmatrix} \text{color of strand 1} \\ \text{color of strand 2} \\ \text{color of strand 3} \\ \text{color of strand 4} \end{pmatrix}$$

subject to the equations from the crossings

Fig 8



coloring $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \in \underbrace{\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z} \times \cdots \times \mathbb{Z}/k\mathbb{Z}}_{\#\text{ of strands}}$

<u>On HW</u>

- Sum of 2 k-colorings is a k-coloring
- If $\ell \in \mathbb{Z}/k\mathbb{Z}$ and $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$ is a k-coloring then so is $\ell \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \ell c_1 \\ \ell c_2 \\ \ell c_3 \\ \ell c_4 \end{pmatrix}$

So K-colorings of a diagram form a vector space (if K is prime)

or "module" (if K is not prime)

Neat Application

Thm The set of K-colorings of a diagram D has # of elements a power of K

Idea If V is the set of colorings we can find a basis of b elements for V and b/c we work mod K if c is a coloring

$\in \mathbb{Z}/K\mathbb{Z}$

$$c = \alpha_1 b_1 + \cdots + \alpha_b b_b$$

basis vectors

$\Rightarrow$ there are k^b choices for a coloring in V.

$$c = \alpha_1 b_1 + \dots + \alpha_b b_b$$

$\square$

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* Use linear algebra to determine existence of k-colorings

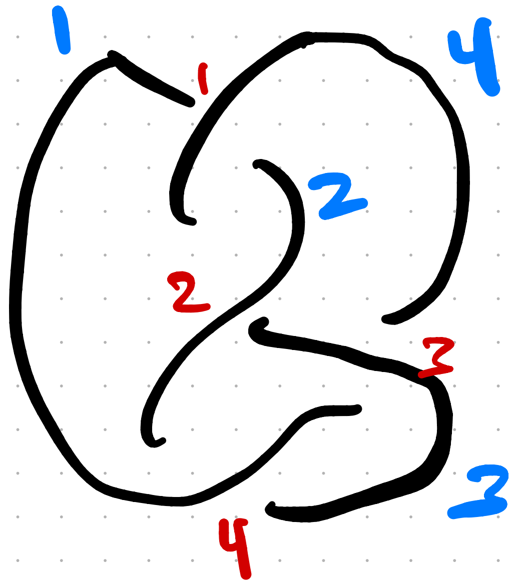
Reminder of determinants

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$\det\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} =$$ alternating sum of 3×3 determinants

Ex



x_i = color of ith strand

1 $\quad 2x_4 - x_1 - x_2 = 0$

2 $\quad 2x_2 - x_4 - x_3 = 0$

3 $\quad 2x_3 - x_4 - x_1 = 0$

4 $\quad 2x_1 - x_2 - x_3 = 0$

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \\ 2 & -1 & -1 & 0 \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \\ 2 & -1 & -1 & 0 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

ARC-CROSSING MATRIX

$\hat{M}$

WANT SOLUTIONS

We'll have solutions if $\det \hat{M} = 0 \pmod{K}$

Always works!

How about non-constant solutions?

$$\underbrace{\begin{pmatrix} -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \\ 2 & -1 & -1 & 0 \end{pmatrix}}_{\hat{M}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

M

Assumes D is the diagram of a knot.

- Remove any row & any column from $\hat{M}$ to get M

Thm D has a non-trivial k-coloring iff $\det M = 0 \mod k$

$$\det \overbrace{\begin{pmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix}}^{M}$$

$$= -1 \det\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} - (-1)\det\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

$$+ 0 \cdot \det\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

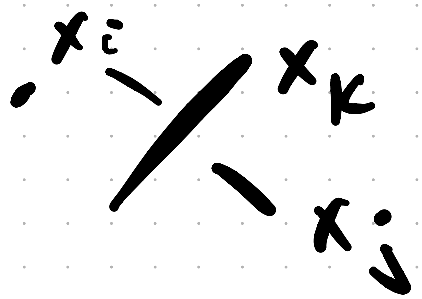
$$= -1(4) + 1(-1)$$

$$= -5$$

$= 0 \bmod 5$

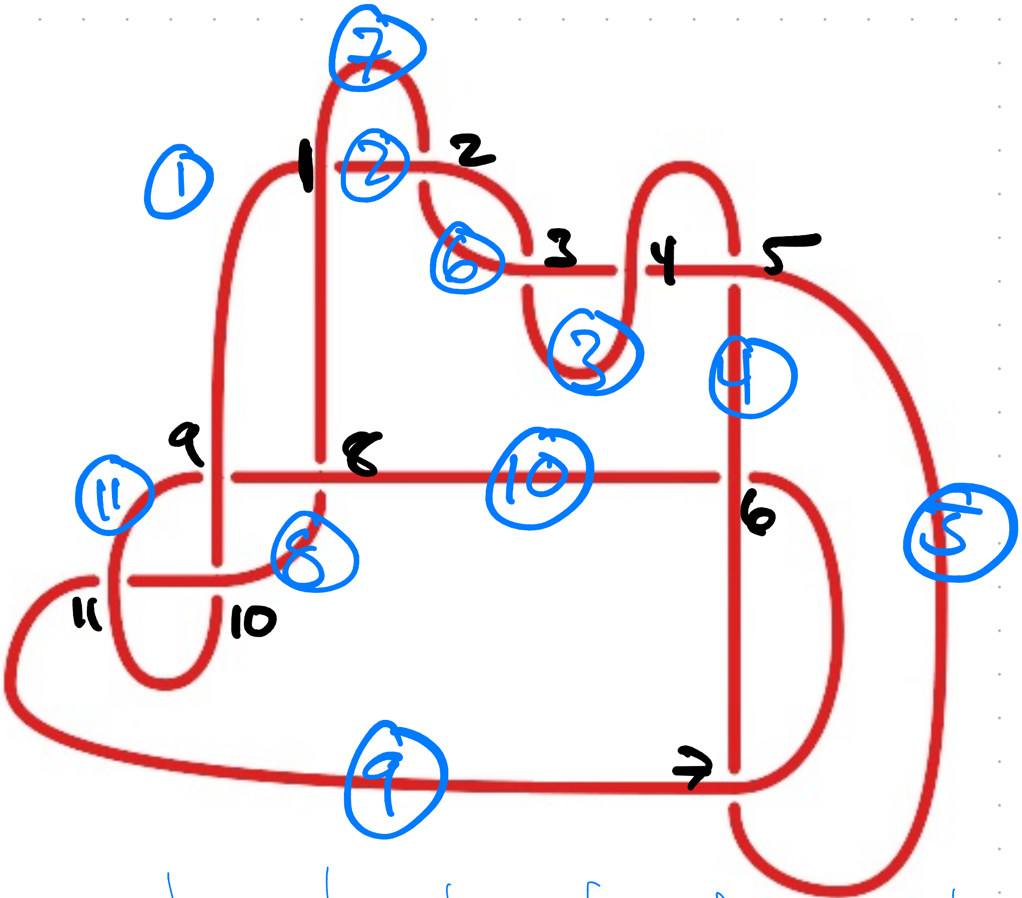
so Fig 8 Knot has a 5-coloring but no other k-coloring

To calculate determinant of a diagram

- variable for each strand
- 

  $2x_k - x_i - x_j = 0$

- Form the crossing-arc matrix $\widetilde{M}$
- For a diagram of knots remove a row and a column from $\hat{M}$ to get M
- D can be k-colored iff $\det \hat{M} = 0 \mod k$.



	1	2	3	4	5	6	7	8	9	10	11	
1	−1	−1					2					
2		2				−1	−1					
3		−1	−1			2						
4				2	−1	−1						
5				−1	−1	2						
6					2				−1	−1		
7					−1	−1			2			
8								−1	−1		2	
9		2										
10		−1									−1	−1
11									2			−1
									−1	−1		2

	1	2	3	4	5	6	7	8	9	10	11
1	−1	−1					2				
2		2				−1	−1				
3		−1	−1			2					
4			2		−1	−1					
5			−1	−1	2						
6				2					−1	−1	
7				−1	−1				2		
8							−1	−1		2	
9	2									−1	−1
10	−1							2			−1
11								−1	−1		2

$$M = \begin{array}{c} \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}$$

If we did this right $\det M = 61$.

$\square$