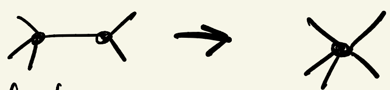
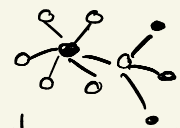
Thm If D is a connected, reduced alternating diagram of a link L then $\operatorname{span} X(L) = 4c(D)$ where $c(D)$ is the number of crossings of D.

Lemma 1 (Euler's Thm)
If G is a connected, planar graph (nonempty) then $V - E + F = 2$ where V = # vertices, E = # edges, F = # regions, including the outer one.

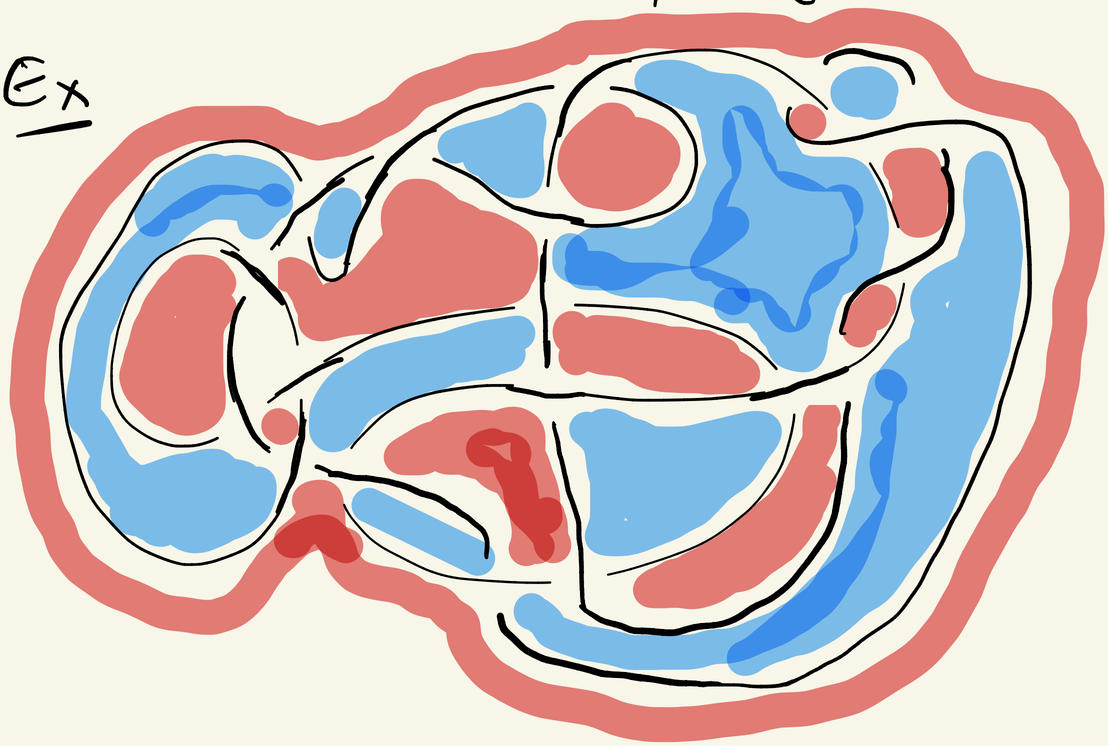
pf Induct on E and shrink an edge e.g.  to do the induction. □

Lemma 2 Every connected graph s.t. every cycle has an even # of edges is bipartite (i.e. vertices can be colored B & W s.t. adjacent vertices are different colors.

pf Color the first vertex B, its neighbors W and so on.
if it weren't  a bipartite coloring, there would be an odd cycle. □

Proposition Let D be a connected link diagram. Then it can be checkerboard colored & the total number of regions is $2 + c(D)$.

Ex



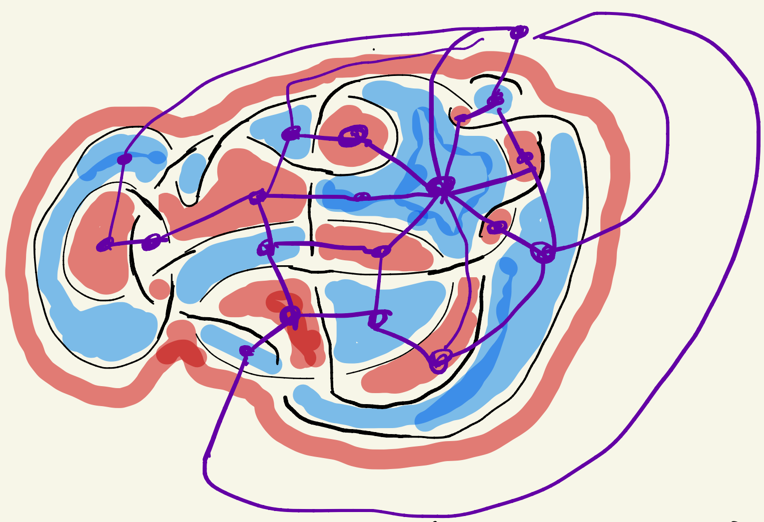
<u>Proposition</u> let D be a connected link diagram. Then it can be checkerboard colored & the total number of regions is $2 + c(D)$.

<u>pf</u> let G be the shadow of the knot. (i.e. D, but forget the crossing info.) Notice each vertex of G has degree 4. The total # of edges is then

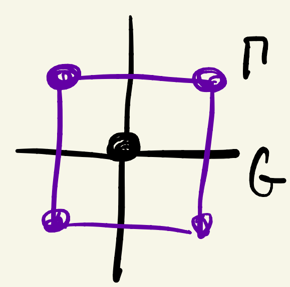
$\frac{1}{2}(4c(D)) = 2c(D)$ and the total # of vertices is $c(D)$. By Euler's Theorem

$$c(D) - 2c(D) + F = 2 \implies F = 2 + c(D).$$

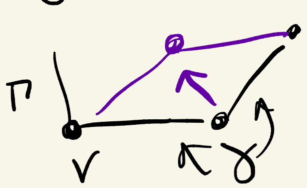
let Γ be the dual graph to G:



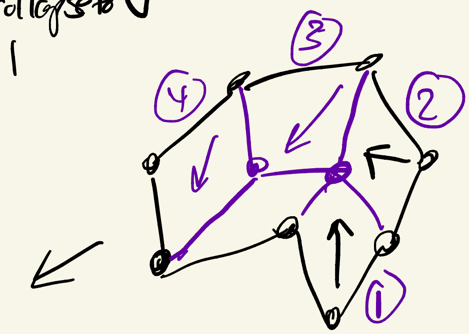
Around each vertex of G we have



If $\gamma \subset \Gamma$ is a cycle mark a vertex v



use this move to collapse inward until we collapse to v
$\implies$ even # of edges in Γ.



Thus Γ is bipartite which is what we were to prove. $\square$

proof on Main Theorem

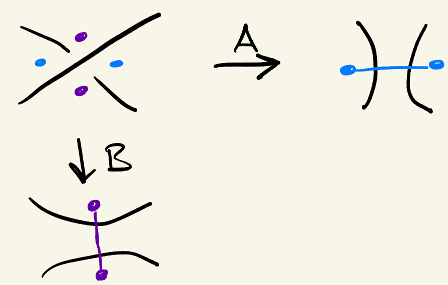
Since $X(K) = (A^{+3})^{-w(D)} \langle D \rangle$

the span of $X(K)$ is equal to the span of $\langle D \rangle$.

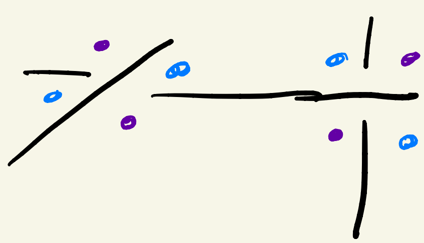
Recall $$\langle D \rangle = \sum_{S} A^{a(S)} A^{-b(S)} (-A^{2}-A^{-2})^{|S|-1}$$

where S is a choice ("state") of A smoothing or B smoothing at each crossing of D & $|S|$ is the number of circles in S.

Recall that the smoothings are

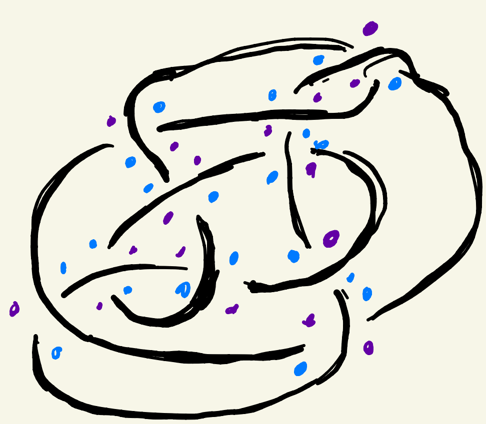


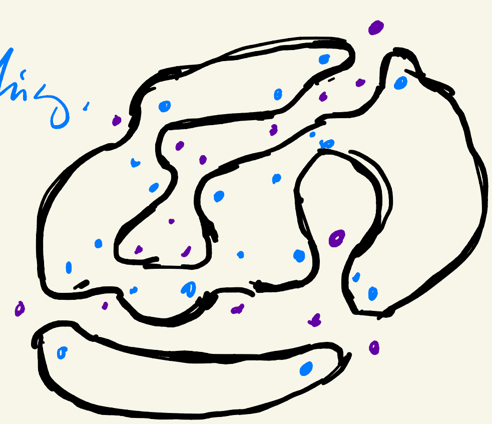
Since D is alternating in each region we have along each edge.



So each circle in the all A-smoothing contains the blue dots and each circle in all-B smoothing contains purple dots

eg.

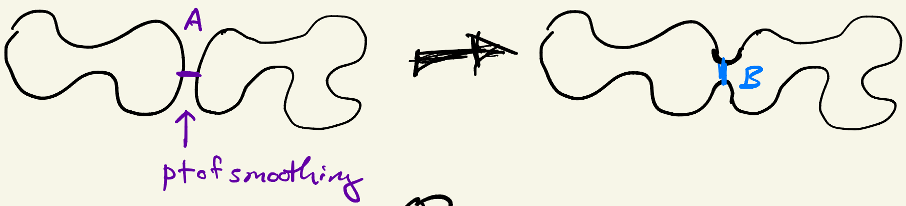




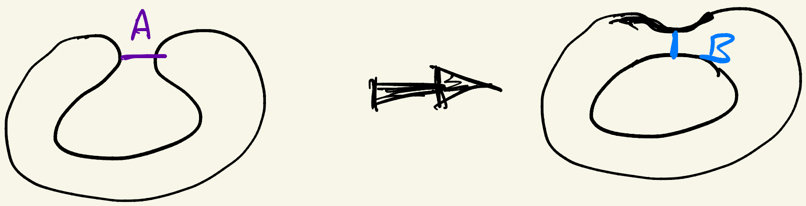
We conclude that

$$\begin{pmatrix}\text{\# circles in} \\ \text{all } A\text{-smoothing}\end{pmatrix} + \begin{pmatrix}\text{\# circles in} \\ \text{all } B\text{-smoothing}\end{pmatrix} = \#\text{ regions} = c(D) + 2.$$

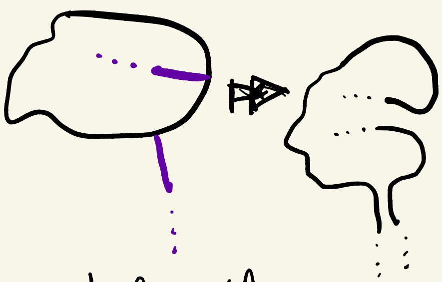
Also notice that if S_1 is a state (choice of A or B smoothing at each crossing) w/ $|S_1| = \#$ circles and if S_2 is obtained by switching one A smoothing to one B smoothing then $|S_2| = |S_1| \pm 1$ b/c either two circles merge or one circle splits (fusion or fission)



OR

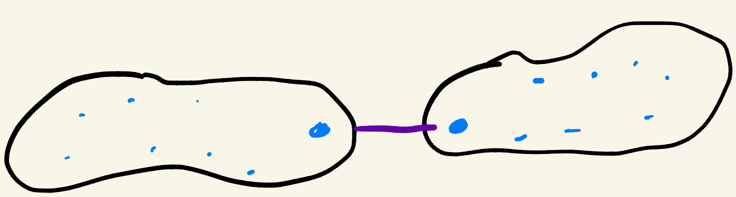


(The Jordan Curve Theorem implies we can't have



which would mean the # of circles didn't change)

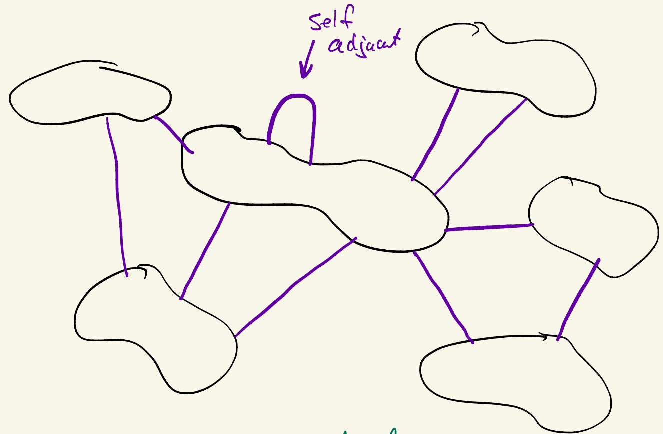
Now for the all A-splitting each bridge joins distinct circles b/c otherwise



the diagram wouldn't be reduced

→

Observe if we had a bridge that was self adjacent we could draw a circle containing the bridge & disjoint from the rest of the diagram
$\Rightarrow$ diagram not reduced.



Now consider the highest power of A arising from a state S_1: The term S_1 contributes to $\langle S_1 \rangle$ is $A^{a(S_1)} A^{-b(S_1)} (-A^2 - A^{-2})^{|S_1|-1}$

& the highest power of A here is

$$a(S_1) - b(S_1) + 2(|S_1| - 1)$$

if S_2 is obtained by changing an A smoothing to a B smoothing, then the highest power for its term is

$$a(S_1) - 1 - (b(S_1) + 1) + 2(|S_1| \pm 1 - 1)$$

$$\leq a(S_1) - b(S_1) + 2(|S_1| - 1)$$

w/ equality only if the A to B switch was a fission.

Thus the all A-smoothing contributes the highest power of A to $\langle D \rangle$ and it is <u>unique</u> in doing so.

from $(-A^2 - A^{-2})^{|S_B|-1}$

Similarly the all B-smoothing uniquely contributes the <u>lowest</u> power of A to $\langle D \rangle$

$\Rightarrow$ If we call S_A the all A-smoothing and S_B the all B-smoothing then

$$
\begin{aligned}
\operatorname{span}\langle D \rangle &= a(S_A) - b(S_A) + 2(|S_A| - 1) \\
&\quad - \big(a(S_B) - b(S_B) - 2(|S_B| - 1)\big) \\
&= c(D) - 0 + 2|S_A| - 2 \\
&\quad + 0 + c(D) + 2|S_B| - 2 \\
&= 2c(D) + 2(|S_A| + |S_B| - 2) \\
&\overset{\text{lemma 1}}{=} 2c(D) + 2(c(D) + 2 - 2) \\
&= 4c(D).
\end{aligned}
$$

$\square$

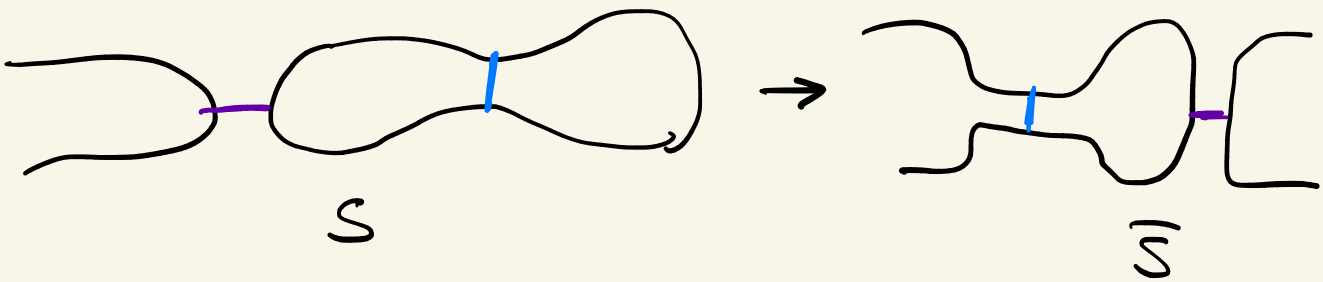
See next page for more!

Thm If D is a connected non-alternating diagram of a link L then

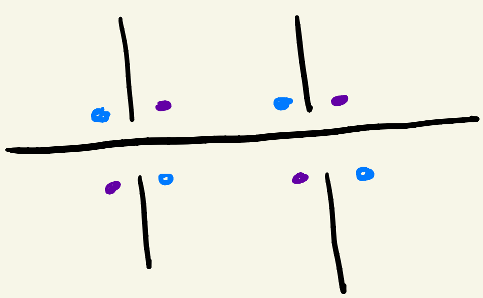
$$\operatorname{span} X(L) < 4c(D)$$

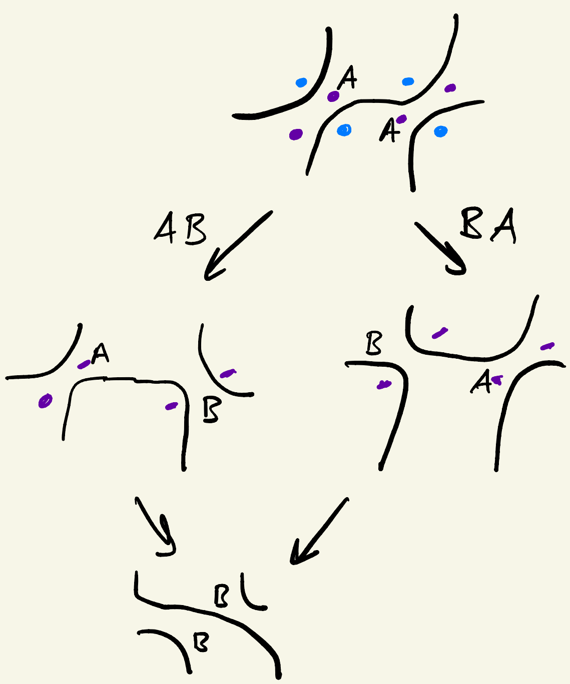
pf We continue the proof from before. Consider a state S. The <u>dual state</u> $\bar{S}$ to S is the one obtained by changing all A-smoothings in S to B smoothings and all B-smoothings to A-smoothings.

eg



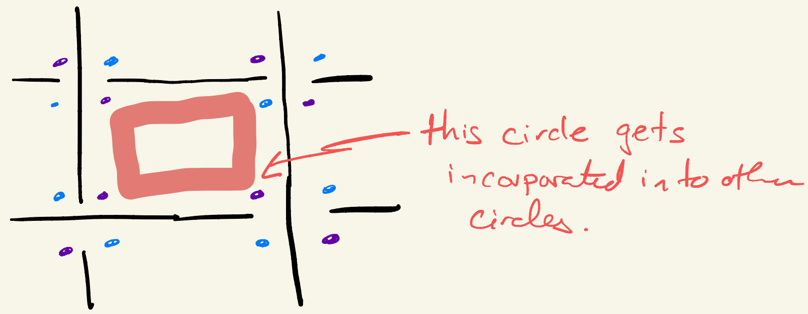
Focus on the spot where we have the non alt. crossing.





We see that in both cases we have nested circles.

Since the boundaries of the regions of the knot diagram get broken into pieces to make up the circles we must have fewer circles than regions eg



Thus, $|S| + |\bar{S}| < \#\text{regions} = c(D) + 2$

Thus, $\max\deg\langle D\rangle \leq a(S) - b(S) + 2|S| - 2$

$$\min\deg\langle D\rangle \geq a(\bar{S}) - b(\bar{S}) + 2|\bar{S}| - 2$$
$$= b(S) - a(S) + 2|\bar{S}| - 2$$

$$\Rightarrow \operatorname{span}\langle D\rangle \leq 2a(S) - 2b(S) + 2(|S| + |\bar{S}| - 2)$$
$$\leq 2c(D) + 2(|S| + |\bar{S}| - 2)$$
$$< 2c(D) + 2(c(D) + 2 - 2)$$
$$= 4c(D)$$

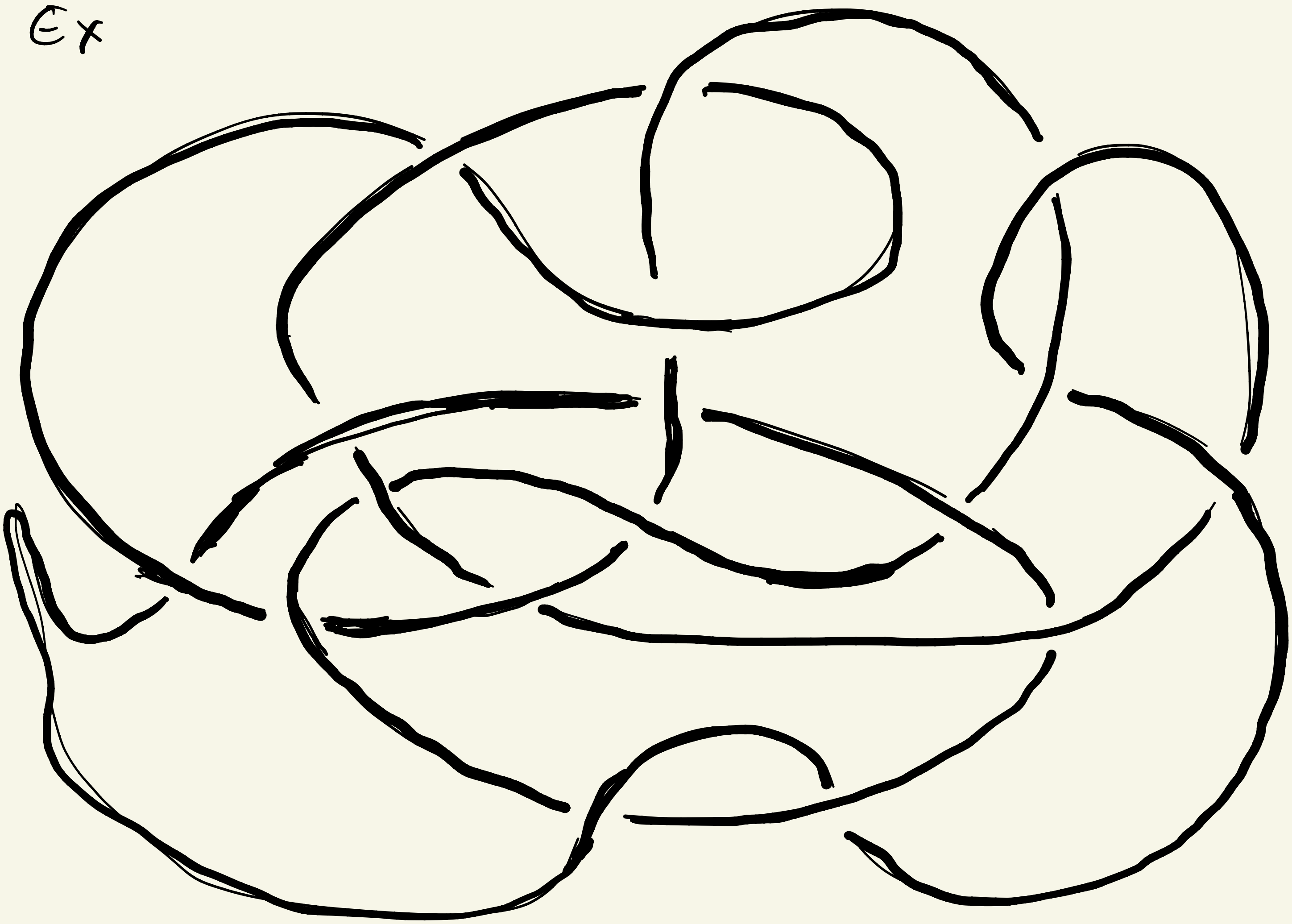
the key!

$\square$

<u>Corollary</u> (Tait Conjectures)

The crossing # for an alternating knot is achieved in every reduced alternating diagram.

Ex



This is not the unknot! Indeed it cannot be drawn w/ fewer crossings.